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DIGITAL CONTENT MATERIAL

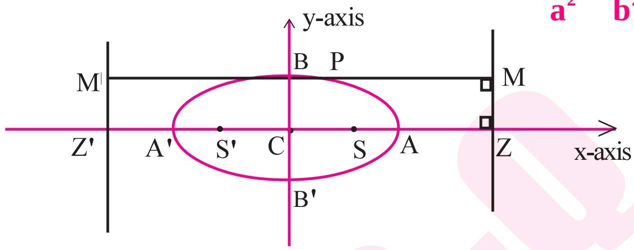
ELLIPSE - INDEX

4. ELLIPSE

SYNOPSIS POINTS

Def: The locus of a point in a plane, which moves such that its distance from a fixed point (focus) bears a constant ratio e, $0 < e < 1$, to its distance from a fixed line (directrix) is called an ellipse.

I. The following terminology holds true w.r.to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, a > b:



1. The **eccentricity** of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\mathbf{e=\frac{\sqrt{a^2-b^2}}{a}}$ $\because b^2=a^2(1-e^2)$. Also

   $\mathbf{a^2e^2=a^2-b^2}$

2. The **foci** of the ellipse are **S=(ae, 0), S'(−ae,0)** i.e., $(\pm\sqrt{a^2-b^2},0)$

3. The **feet of the directrices** are $\mathbf{Z=\left(\frac{a}{e},0\right)}$, $\mathbf{Z'=\left(-\frac{a}{e},0\right)}$ & the **equation** of the directrices is $\mathbf{x=\pm\frac{a}{e}}$

4.1. **AA'** is called the **major axis**, **BB'** is called the **minor axis** of the ellipse

4.2. The **length** of the **major axis** is **2a** and the **equation** of the **major axis** is **y=0**

The **length** of **minor axis** is **2b** and the **equation** of the **minor axis** is **x=0**

5.1. **A(a,0), A'(-a,0)** are called the **vertices of the ellipse.**

B(0,b), B'(0, -b) are called the **vertices on the minor axis.**

5.2. The **equation** of the **tangents** at the **vertices** is **x=±a**

6.1. The **equation** of the **latus recta** is **x=±ae**

6.2. The **ends** of the **latusrecta** are $\mathbf{\left(ae,\pm\frac{b^2}{a}\right)}$ **and** $\mathbf{\left(-ae,\pm\frac{b^2}{a}\right)}$

6.3. The **length** of the **latus rectum** is $\mathbf{\frac{2b^2}{a}}$

7. The **focal distance** of the point $\mathbf{P(x_1,y_1)}$ on the ellipse w.r.t the focus **S** is $\mathbf{SP=a-ex_1}$ and the focal distance of the point **P** on the ellipse w.r.t the focus **S' is** $\mathbf{S'P=a+ex_1}$

8. The equation of the **auxiliary circle** of the ellipse is $x^2+y^2=a^2$

9. The equation of the **director circle** of the ellipse is $x^2+y^2=a^2+b^2$

II. *The following results hold true w.r.to the horizontal ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$:

1. **Notation:** $S=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$; $S_1=\frac{x_1x}{a^2}+\frac{y_1y}{b^2}-1$; $S_{11}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$; $S_{12}=\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}-1$
2. **Relative positions** of a point $P(x_1,y_1)$ and the ellipse S=0
   (i) The point $P(x_1, y_1)$ lies on the ellipse $S=0 \Leftrightarrow S_{11}=0$
   (ii) The point $P(x_1, y_1)$ lies inside the ellipse $S=0 \Leftrightarrow S_{11}<0$
   (iii) The point $P(x_1, y_1)$ lies outside the ellipse $S=0 \Leftrightarrow S_{11}>0$
3. The equation of the chord joining the points $A(x_1, y_1)$ & $B(x_2,y_2)$ on the ellipse S=0 is $S_1+S_2=S_{12}$
4. The equation of the tangent at $P(x_1, y_1)$ on the ellipse S=0 is $S_1=0$
5. The equation of the normal at $P(x_1, y_1)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$
6. The condition for the line y=mx+c to be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $c^2=a^2m^2+b^2$.
7. The equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ having slope m is $y=mx\pm\sqrt{a^2m^2+b^2}$
8. Two tangents can be drawn from an external point $P(x_1,y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ &

   if m_1,m_2 are the slopes of the two tangents then $m_1+m_2=\frac{2x_1y_1}{x_1^2-a^2}$, $m_1m_2=\frac{y_1^2-b^2}{x_1^2-a^2}$
9. The combined equation of the pair of tangents drawn from an external point $P(x_1,y_1)$ to the ellipse $S=0$ is $S_1^2=S_{11}S$.
10. The equation of the chord of contact of $P(x_1,y_1)$ w.r.to the ellipse S=0 is $S_1=0$
11. The equation of the chord of the ellipse S=0 having $P(x_1,y_1)$ as its midpoint is $S_1=S_{11}$.

***III. Parametric treatment*:**

12. The parametric point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $P(a\cos\theta, b\sin\theta)$ and is simply denoted by θ.
13. The equation of the tangent at $P(\theta)$ on the ellipse S=0 is $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$
14. The equation of the normal at $P(\theta)$ on the ellipse S=0 is $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$

ILLUSTRATIVE RESULTS WITH CERTAIN PROOFS

1.The following notation is adapted throughout this chapter.

$S=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$	$S=ax^2+by^2-ab$
$S_1=\frac{x_1x}{a^2}+\frac{y_1y}{b^2}-1$	$S_1=ax_1x+by_1y-ab$
$S_{11}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$	$S_{11}=ax_1^2+by_1^2-ab$
$S_{12}=\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}-1$	$S_{12}=ax_1x_2+by_1y_2-ab$

2. Relative positions of a point $P(x_1,y_1)$ and the ellipse $S=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$

(i) The point $P(x_1,y_1)$ lies on the ellipse $S=0 \Leftrightarrow S_{11}=0$

(ii) The point $P(x_1,y_1)$ lies inside the ellipse $S=0 \Leftrightarrow S_{11}<0$

(iii) The point $P(x_1,y_1)$ lies outside the ellipse $S=0 \Leftrightarrow S_{11}>0$

Ex: Determine the relative positions of the following points w.r.t the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$

(i) (5, 4) (ii) (5,0)(iii) (0,0) (iv) (1,4)

Sol: (i) S_{11} is just the substitution of (x_1,y_1) in $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$

substituting (5,4) in $\frac{x^2}{25}+\frac{y^2}{16}-1$, we get $\frac{5^2}{25}+\frac{4^2}{16}-1=1+1-1=1>0$

$\therefore$ (5,4) lies outside the given ellipse.

(ii) Substituting (5,0) in $\frac{x^2}{25}+\frac{y^2}{16}-1$, we get $\frac{5^2}{25}+\frac{0}{16}-1=1-1=0$

$\therefore$ (5,0) lies on the given ellipse.

(iii) Substituting (0,0) in $\frac{x^2}{25}+\frac{y^2}{16}-1$, we get $\frac{0}{25}+\frac{0}{16}-1=-1<0$

$\therefore$ (0,0) lies inside the given ellipse.

(iv) Substituting (1,4) in $\frac{x^2}{25}+\frac{y^2}{16}-1$, we get $\frac{1}{25}+\frac{16}{16}-1=\frac{1}{25}>0$

$\therefore$ (1,4) lies outside the given ellipse.

3. Theorem: The equation of the chord joining the points $A(x_1,y_1)$ & $B(x_2,y_2)$ on the ellipse $S=0$ is $S_1+S_2=S_{12}$.

Proof: Let $S=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ be the given ellipse.

$A(x_1,y_1)$ & $B(x_2,y_2)$ are two points on the ellipse $S=0 \Rightarrow S_{11}=0, S_{22}=0$

Now, consider the equation $S_1+S_2=S_{12}$ .....(1)

$$\Rightarrow \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1+\frac{xx_2}{a^2}+\frac{yy_2}{b^2}-1=\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}-1 \text{..........(2)}$$

It is a first degree equation in x and y and hence it represents a straight line.

Substituting $A(x_1,y_1)$ in equation (2), we get $S_{11}+S_{12}=S_{12}$ which is true, since $S_{11}=0$

$\therefore$ A lies on the line (1)

Substituting $B(x_2,y_2)$ in equation (2), we get $S_{12}+S_{22}=S_{12}$ which is true, since $S_{22}=0$

$\therefore$ B lies on the line (1)

Here, A,B both satisfies the linear equation $S_1+S_2=S_{12}$

$\therefore$ The equation of $\overline{AB}$ is $S_1+S_2=S_{12}$.

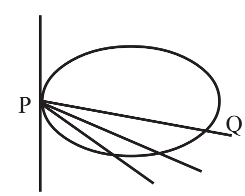
4. Theorem: The equation of the tangent at $P(x_1,y_1)$ on the ellipse $S=0$ is $S_1=0$

Proof: $P(x_1,y_1)$ is a point on the ellipse $S=0 \Rightarrow S_{11}=0$

Let $Q(x_2,y_2)$ be a point on the ellipse $S=0$

The equation of the chord joining P,Q is $S_1+S_2=S_{12}$.

If Q approaches to P then the chord PQ becomes the tangent at P.



$\therefore$ The equation of the tangent at P is $\underset{Q\to P}{\text{Lt}}\{S_1+S_2=S_{12}\}$

$\Rightarrow S_1+S_1=S_{11} \Rightarrow 2S_1=0 \Rightarrow S_1=0$ ($\because S_{11}=0$)

i.e., $\frac{x_1x}{a^2}+\frac{y_1y}{b^2}-1=0$

Note: The slope of the tangent $\frac{x_1x}{a^2}+\frac{y_1y}{b^2}-1=0$ is $-\frac{b^2}{a^2}\frac{x_1}{y_1}$

Ex: Find the equation of the tangent at (3,2) on $\frac{x^2}{18}+\frac{y^2}{8}=1$ (or) $8x^2+18y^2=144$

Sol: The equation of the tangent at (x_1,y_1) on $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ is $S_1=\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=0$

$\therefore$ The equation of the tangent at (3,2) on $\frac{x^2}{18}+\frac{y^2}{8}=1$ is $\frac{3x}{18}+\frac{2y}{8}=1 \Rightarrow \frac{x}{6}+\frac{y}{4}=1$

i.e., $2x+3y-12=0$

OR

The equation of the tangent at (x_1,y_1) on $ax^2+by^2=ab$ is $ax_1x+by_1y-ab=0$

The equation of the tangent at (3,2) on $8x^2+18y^2=144$ is $8(3)(x)+18(2)y-144=0$

$\Rightarrow 24x+36y-144=0 \Rightarrow 2x+3y-12=0$

5. Theorem: The equation of the normal at $P(x_1,y_1)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$

Proof: We know that the slope of the tangent $\frac{x_1x}{a^2}+\frac{y_1y}{b^2}-1=0$ at (x_1,y_1) on S=0 is $\frac{-b^2x_1}{a^2y_1}$

$\Rightarrow$ The slope of the normal at (x_1,y_1) is $\frac{a^2y_1}{b^2x_1}$

$\therefore$ The equation of the normal at (x_1,y_1) with slope $\frac{a^2y_1}{b^2x_1}$ is

$$y-y_1=\frac{a^2y_1}{b^2x_1}(x-x_1)\Rightarrow b^2x_1(y-y_1)=a^2y_1(x-x_1)$$

$$\Rightarrow b^2x_1y-b^2x_1y_1=a^2y_1x-a^2x_1y_1\Rightarrow a^2y_1x-b^2x_1y=a^2x_1y_1-b^2x_1y_1$$

$$\Rightarrow \frac{a^2y_1x}{x_1y_1}-\frac{b^2x_1y}{x_1y_1}=a^2-b^2\Rightarrow \frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$$

Ex: Find the equation of the normal at (3,2) on $\frac{x^2}{9}+\frac{y^2}{4}=2$

Sol: The equation of the given ellipse is $\frac{x^2}{18}+\frac{y^2}{8}=1\Rightarrow a^2=18, b^2=8$

Also $(x_1,y_1)=(3,2)$

$\therefore$ the equation of the normal is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$

$$\Rightarrow \frac{18x}{3}-\frac{8y}{2}=18-8\Rightarrow 6x-4y=10\Rightarrow 3x-2y=5$$

6. Theorem: The condition for the line y=mx+c to be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $c^2=a^2m^2+b^2$.

Proof: Suppose y=mx+c .... (1) is a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Let $P(x_1,y_1)$ be the point of contact of the given line y=mx+c and the ellipse

Now, the equation of the tangent at P on S=0 is $S_1=0 \Rightarrow \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=0$.........(2)

Now, (1) and (2) represent the same line $\Rightarrow$ corresponding coefficients are proportional

$$\Rightarrow \frac{x_1}{a^2m}=\frac{y_1}{b^2(-1)}=\frac{-1}{c}\Rightarrow x_1=\frac{-a^2m}{c}, y_1=\frac{b^2}{c}$$

But $P(x_1,y_1)$ lies on the line y=mx+c $\Rightarrow y_1=mx_1+c$

$$\Rightarrow \frac{b^2}{c}=m\left(\frac{-a^2m}{c}\right)+c\Rightarrow b^2=-a^2m^2+c^2\Rightarrow c^2=a^2m^2+b^2$$

Corollary: Show that the condition for the line $lx+my+n=0$ to be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $n^2=a^2l^2+b^2m^2$

Sol: Given line is $lx+my+n=0 \Rightarrow my=-lx-n \Rightarrow y=\left(-\frac{l}{m}\right)x+\left(-\frac{n}{m}\right)$

The line is a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \left(-\frac{n}{m}\right)^2=a^2\left(-\frac{l}{m}\right)^2+b^2$

$\Rightarrow n^2=a^2l^2+b^2m^2$.

Ex: Find the point of contact of the tangent $x-y+5=0$ and the ellipse $9x^2+16y^2=144$.

Sol: The equation of the tangent is $x-y+5=0 \Rightarrow y=x+5 \Rightarrow m=1, c=5$

The given ellipse is $9x^2+16y^2=144 \Rightarrow \frac{9x^2}{144}+\frac{16y^2}{144}=1 \Rightarrow \frac{x^2}{16}+\frac{y^2}{9}=1 \Rightarrow a^2=16, b^2=9$

Now, the point of contact is $\left(\frac{-ma^2}{c}, \frac{b^2}{c}\right)=\left(\frac{-1(16)}{5}, \frac{9}{5}\right)=\left(-\frac{16}{5}, \frac{9}{5}\right)$

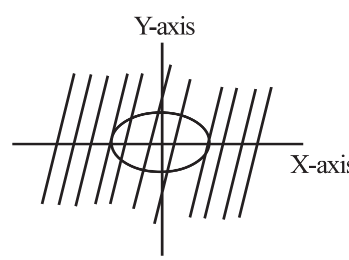
7. Theorem: Prove that the equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ having slope m is $y=mx\pm\sqrt{a^2m^2+b^2}$

Proof: Consider the line $y=mx+c$ and the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

The equation $y=mx+c$ represents a family of parallel lines for the arbitrary constant c and for fixed m .

Among these only 2 lines touch the ellipse satisfying the tangential condition

$c^2=a^2m^2+b^2$ i.e., $c=\pm\sqrt{a^2m^2+b^2}$



$\therefore$ the equation of the tangents with slope m to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $y=mx\pm\sqrt{a^2m^2+b^2}$

Ex: Find the equation of the tangents to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ which are parallel to $x-y+2=0$

Sol: The equation of the given line is $x-y+2=0 \Rightarrow y=x+1 \Rightarrow m=1$

The equation of the ellipse is $\frac{x^2}{16}+\frac{y^2}{9}=1 \Rightarrow a^2=16 \ \& \ b^2=9$

$\therefore$ the equation of the required tangents is $y=mx\pm\sqrt{a^2m^2+b^2}$

i.e., $y=(1)x\pm\sqrt{16(1^2)+9} \Rightarrow y=x\pm\sqrt{25} \Rightarrow y=x\pm5$

$\therefore$ the equation of the two tangents are $x-y\pm5=0$

8. Theorem: Show that two tangents can be drawn from an external point $P(x_1,y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ & hence show that , if m_1,m_2 are the slopes of the two tangents then $m_1+m_2=\frac{2x_1y_1}{x_1^2-a^2}$, $m_1m_2=\frac{y_1^2-b^2}{x_1^2-a^2}$

Proof: Given that $P(x_1,y_1)$ is an external point to the ellipse $S=0 \Rightarrow S_{11}>0$

Let $y=mx\pm\sqrt{a^2m^2+b^2}$ be a tangent to the ellipse with slope m

If it pass through $P(x_1,y_1)$, then $y_1=mx_1\pm\sqrt{a^2m^2+b^2} \Rightarrow y_1-mx_1=\pm\sqrt{a^2m^2+b^2}$

$\Rightarrow (y_1-mx_1)^2=a^2m^2+b^2 \Rightarrow y_1^2+m^2x_1^2-2x_1y_1m=a^2m^2+b^2$

$\Rightarrow (x_1^2-a^2)m^2-2x_1y_1m+(y_1^2-b^2)=0$.........(1)

It is a quadratic equation in m

Now, the discriminant of (1) is $\Delta=b^2-4ac=(-2x_1y_1)^2-4(x_1^2-a^2)(y_1^2-b^2)$

$=4x_1^2y_1^2-4(x_1^2y_1^2-x_1^2b^2-y_1^2a^2+a^2b^2)=4(b^2x_1^2+a^2y_1^2-a^2b^2)=4a^2b^2\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)>0$ $[\because S_{11}>0]$

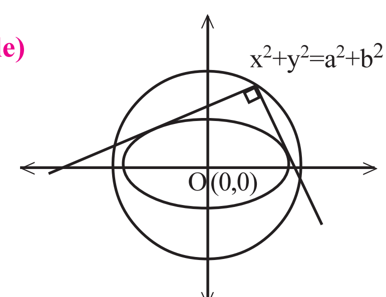
$\therefore$ the two values of m are real and different.

Hence there exists two tangents from an external point $P(x_1,y_1)$ to the ellipse.

We take m_1,m_2 as the roots (1), which are nothing but the slopes of the two tangents.

Sum of the roots $m_1+m_2=\frac{2x_1y_1}{x_1^2-a^2}$, product of the roots $m_1m_2=\frac{y_1^2-b^2}{x_1^2-a^2}$

Cor1: The locus of point of intersection of perpendicular tangents to the ellipse $S=\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $x^2+y^2=a^2+b^2$. (Director circle)



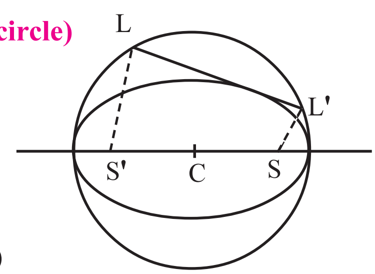
Proof : Let $P(x_1,y_1)$ be any point on the locus

Let m_1,m_2 be the slopes of the tangents to the ellipse which intersect at right angle $\Rightarrow m_1m_2=-1$

$\Rightarrow \frac{y_1^2-b^2}{x_1^2-a^2}=-1 \Rightarrow y_1^2-b^2=-(x_1^2-a^2)=-x_1^2+a^2$

$\Rightarrow x_1^2+y_1^2=a^2+b^2$ $\therefore$ the locus of $P(x_1,y_1)$ is $x^2+y^2=a^2+b^2$.

Cor 2:The locus of feet of the perpendiculars drawn from the foci, to any tangent of of the ellipse $S=\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $x^2+y^2=a^2$. (Auxiliary circle)



Proof: The equation of any tangent with slope m to the ellipse is

$y=mx\pm\sqrt{a^2m^2+b^2} \Rightarrow y-mx=\pm\sqrt{a^2m^2+b^2}$ ...(1)

The equation to the perpendicular from either focus $(\pm ae,0)$

on to this tangent is $y-0=-\frac{1}{m}(x\pm ae) \Rightarrow my+x=\pm ae$ ...(2)

$(1)^2+(2)^2 \Rightarrow (y-mx)^2+(my+x)^2=a^2m^2+b^2+(ae)^2 \Rightarrow x^2(1+m^2)+y^2(1+m^2)=a^2m^2+a^2(1-e^2)+a^2e^2$.

$\Rightarrow (x^2+y^2)(1+m^2)=a^2(m^2+1) \Rightarrow x^2+y^2=a^2$.

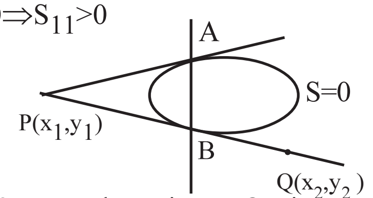
9.Theorem: The equation of the chord of contact of $P(x_1,y_1)$ w.r.to the ellipse S=0 is $S_1=0$

Ex: Find the equation of the chord of contact of (1,-2) w.r.t the ellipse $4x^2+5y^2=20$

Sol: The equation of the chord of contact of (x_1,y_1) w.r.t $S=ax^2+by^2=ab$ is $S_1=ax_1x+by_1y-ab=0$
$\Rightarrow 4(1)x+5(-2)y-20 \Rightarrow 4x-10y-20=0 \Rightarrow 2x-5y-10=0$

10. Theorem: The combined equation of the pair of tangents drawn from an external point $P(x_1,y_1)$ to the ellipse S=0 is $S_1^2=S_{11}S$.

Proof: Given that $P(x_1,y_1)$ is an external point, to the ellipse $S=0 \Rightarrow S_{11}>0$
Let AB be the chord of contact from $P(x_1,y_1)$ to the ellipse and its equation is $S_1=0$
Let $Q(x_2,y_2)$ be any point on the locus
i.e., $Q(x_2,y_2)$ be a point on the pair of tangents



The ratio that the line AB divides PQ can be determined in 2 ways: the ratio PB:QB is equal to (i) the ratio PB : QB is equal to $\sqrt{S_{11}}:\sqrt{S_{22}}$
(ii) the ratio that the line $S_1=0$ divides the line segment joining the points $P(x_1,y_1),Q(x_2,y_2)$ is

$$-\left(\frac{S_{11}}{S_{12}}\right) \Rightarrow \frac{\sqrt{S_{11}}}{\sqrt{S_{22}}}=-\frac{S_{11}}{S_{12}} \Rightarrow (S_{11})^2(S_{22})=(S_{11})(S_{12})^2 \Rightarrow (S_{11})S_{22}=(S_{12})^2$$

Hence the locus of $Q(x_2,y_2)$ is $S_{11}S=S_1^2$

Ex: Find the combined equation of the pair of tangents drawn from (1,2) to the ellipse $x^2+2y^2=2$

Sol: The equation of the pair of tangents drawn from (x_1,y_1) to the ellipse S=0 is $S_1^2=S_{11}(S)$
$\Rightarrow (1(x)+2(2y)-2)^2=(1^2+2(2^2)-2)(x^2+2y^2-2)$
$\Rightarrow (x+4y-2)^2=(7)(x^2+2y^2-2)$
$\Rightarrow x^2+16y^2+4+8xy-16y-4x=7x^2+14y^2-14$
$\Rightarrow 6x^2-2y^2-8xy+4x+16y-18=0$
$\Rightarrow 3x^2-4xy-y^2+2x+8y-9=0$

11.Theorem:The equation of the chord of the ellipse S=0 having $P(x_1,y_1)$ as its midpoint is $S_1=S_{11}$.

Ex: Find the equation of the chord of the ellipse $2x^2+y^2=4$ having (1,1) as its midpoint.

Sol: The equation of the chord with (x_1,y_1) as midpoint to the ellipse $ax^2+by^2=ab$ is
$S_1=S_{11} \Rightarrow ax_1x+by_1y-ab=ax_1^2+by_1^2-ab$
$\Rightarrow 2(1)x+1(y)-4=2(1^2)+(1)^2-4 \Rightarrow 2x+y=3.$

12.**Theorem: The equation of the polar of the point $P(x_1,y_1)$ w.r.t the ellipse S=0 is $S_1=0$**

Ex: Find the polar of (2,3) w.r.t the ellipse $2x^2+3y^2=6$

Sol: The equation of the polar of (x_1,y_1) w.r.t $S=ax^2+by^2=ab$ is $S=ax_1x+by_1y-ab=0$
$\therefore$ The polar of (2,3) w.r.t $2x^2+3y^2=6$ is $2(2)x+3(3)y-6=0 \Rightarrow 4x+9y-6=0$

13.Theorem: The pole of the line lx+my+n=0, (n≠0) w.r.t the ellipse $S=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ is $\left(\frac{-la^2}{n},\frac{-mb^2}{n}\right)$

Eg: Find the pole of the line 2x+3y+1=0 w.r.t the ellipse $\frac{x^2}{3}+\frac{y^2}{2}=1$

Sol: Comparing 2x+3y+1=0 with the equation lx+my+n=0, we get l=2, m=3, n=1.

Comparing $\frac{x^2}{3}+\frac{y^2}{2}=1$ with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, we get $a^2=3$, $b^2=2$.

$\therefore$ Pole $=\left(\frac{-la^2}{n},\frac{-mb^2}{n}\right)=\left(\frac{-3(2)}{1},\frac{-2(3)}{1}\right)=(-6,-6)$

14.Theorem:The condition for the points (x_1,y_1) and (x_2,y_2) to be conjugate w.r.t the ellipse S=0 is $S_{12}=0$

Eg : Show that the points (2,-3), (5,1) are conjugate w.r.t the ellipse $\frac{x^2}{4}+\frac{y^2}{2}=1$

Sol: The condition for the points (x_1,y_1) and (x_2,y_2) to be conjugate w.r.t the ellipse

$S=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ is $S_{12}=\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}-1=0$

Applying the condition $\frac{2(5)}{4}+\frac{(-3)(1)}{2}-1=\frac{10}{4}-\frac{3}{2}-1=\frac{5-3-2}{2}=0$

$\therefore$ the given two points are conjugate w.r.t the given ellipse

15. Theorem: The condition for the lines $l_1x+m_1y+n_1=0$ and $l_2x+m_2y+n_2=0$ to be conjugate w.r.t the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $n_1n_2=a^2l_1l_2+b^2m_1m_2$.

Ex: Find the value k if the lines 4x+3y−6=0 and x−y−k=0 are conjugate w.r.t the ellipse $2x^2+3y^2=6$.

Sol: The equation of the given ellipse is $2x^2+3y^2=6$

$\Rightarrow \frac{2x^2}{6}+\frac{3y^2}{6}=1 \Rightarrow \frac{x^2}{3}+\frac{y^2}{2}=1 \Rightarrow a^2=3,\ b^2=2$

Also $l_1=4$, $m_1=3$, $n_1=-6$ and $l_2=1$, $m_2=-1$, $n_2=-k$

Now, applying the condition $n_1n_2=a^2l_1l_2+bm_1m_2$ we get,

$(-6)(-k)=3(4)(1)+2(3)(-1)$

$\Rightarrow 6k=6 \Rightarrow k=1$

ADDITIONAL QUESTIONS WITH SOLUTIONS

1 **Find the equation of the ellipse in the standard form whose focus is (3,0) & eccentricity is 3/5**

Sol: Given that focus S(ae,0)=(3,0) $\Rightarrow$ ae=3

eccentricity e=3/5 $\Rightarrow$ a(3/5)=3 $\Rightarrow$ a=5

$$\text{Now, } b^2 = a^2(1-e^2) \Rightarrow b^2 = 5^2\left(1-\left(\frac{3}{5}\right)^2\right) = 25\left(\frac{25-9}{25}\right) = 16$$

$\therefore$ The equation of the ellipse in the standard form is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ i.e., $\frac{x^2}{25}+\frac{y^2}{16}=1$

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2 **Find the equation of the ellipse whose focus is (0,1), eccentricity is 2/3 & directrix is 4y–9 = 0**

Sol: Let $P(x_1, y_1)$ be any point on the ellipse

Now, by the focus directrix property of the ellipse we have $\frac{SP}{PM} = e \Rightarrow SP = ePM$

$$\Rightarrow \sqrt{(x_1-0)^2+(y_1-1)^2} = \frac{2}{3}\left|\frac{4y_1-9}{\sqrt{4^2}}\right|$$

$$\Rightarrow x_1^2+(y_1-1)^2 = \frac{1}{(9)(4)}(4y_1-9)^2 \Rightarrow 36(x_1^2+(y_1-1)^2) = (4y_1-9)^2$$

$$\Rightarrow 36(x_1^2+y_1^2-2y_1+1) = 16y_1^2-72y_1+81 \Rightarrow 36x_1^2+36y_1^2-72y_1+36 = 16y_1^2-72y_1+81$$

$$\Rightarrow 36x_1^2+20y_1^2 = 45$$

$\therefore$ The equation of the ellipse is $36x^2+20y^2=45$ (OR)

From the problem, it is clear that the focus lies on the y-axis, & directrix is parallel to the x-axis.

$\therefore$ The required ellipse is a vertical one in the standard form.

Now, focus S(0,be) = (0,1) $\Rightarrow$ be=1

eccentricity e=2/3 $\Rightarrow$ b(2/3)=1 $\Rightarrow$ b=3/2

$$\text{Now, } a^2 = b^2(1-e^2) \Rightarrow a^2 = \left(\frac{3}{2}\right)^2\left(1-\left(\frac{2}{3}\right)^2\right) = \frac{9}{4}\left(\frac{9-4}{9}\right) = \frac{5}{4}$$

$\therefore$ the equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{x^2}{\frac{5}{4}}+\frac{y^2}{\frac{9}{4}}=1 \Rightarrow \frac{4x^2}{5}+\frac{4y^2}{9}=1 \Rightarrow 36x^2+20y^2=45$

3 **Find the equation of the ellipse in the form $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, given that the centre (2, –1), one end of major axis is (2, –5) and $e=\frac{1}{3}$.**

Sol: Given centre, C(h, k) = (2, –1) $\Rightarrow$ h = 2, k = – 1

One end of major axis is B' (2, –5)

$$b = CB' = \sqrt{(2-2)^2+(-5+1)^2}=\sqrt{16}=4, a^2=b^2(1-e^2)=16\left(1-\frac{1}{9}\right)=\frac{128}{9}$$

Equation of the ellipse is $\frac{(x-2)^2}{128/9}+\frac{(y+1)^2}{16}=1\Rightarrow\frac{9(x-2)^2}{128}+\frac{(y+1)^2}{16}=1\Rightarrow 9(x-2)^2+8(y+1)^2=128$

---

4 **Find the equation of the ellipse in the form $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, given that the centre (0, –3), e = 2/3, semi-minor axis = 5**

Sol: Given centre C(h, k) = (0, –3) $\Rightarrow$ h = 0, k = –3, eccentricity e=2/3

Semi minor axis b= 5

Now $b^2=a^2(1-e^2)\Rightarrow b^2=a^2-a^2e^2\Rightarrow 25=a^2-a^2\left(\frac{4}{9}\right)=a^2\left(1-\frac{4}{9}\right)=\frac{5}{9}a^2\Rightarrow a^2=45$

Equation of the ellipse is $\frac{(x-0)^2}{45}+\frac{(y+3)^2}{25}=1\Rightarrow\frac{x^2}{45}+\frac{(y+3)^2}{25}=1$

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5 **Find the equation of the ellipse in the form $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, given that the centre (4, –1), one end of major axis is (–1, –1) and passing through (8, 0).**

Sol: Given centre C(h, k) = (4, –1) $\Rightarrow$ h = 4, k = –1

One end of major axis is A' (–1, –1)

$$a = CA' = \sqrt{(4+1)^2+(-1+1)^2}=\sqrt{25}=5\Rightarrow a=5$$

Given that the ellipse $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ passes through (8, 0)

$$\Rightarrow\frac{(8-4)^2}{25}+\frac{(0+1)^2}{b^2}=1\Rightarrow\frac{1}{b^2}=1-\frac{16}{25}\Rightarrow\frac{1}{b^2}=\frac{9}{25}\Rightarrow b^2=\frac{25}{9}$$

Equation of the ellipse is $\frac{(x-4)^2}{25}+\frac{9(y+1)^2}{25}=1\Rightarrow(x-4)^2+9(y+1)^2=25$

6 **Find the equation of the ellipse in the form $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, given that the Centre(2, –1); $e=\frac{1}{2}$; length of latus rectum 4.**

Sol: Given centre C(h, k) = (2, –1) $\Rightarrow$ h = 2, k = –1, eccentricity $e=\frac{1}{2}$

Given the length of latus rectum $\frac{2b^2}{a}=4 \Rightarrow b^2=2a$ ....(1)

$\Rightarrow a^2(1-e^2)=2a \Rightarrow a^2\left(1-\frac{1}{4}\right)=2a \Rightarrow \frac{3}{4}a=2 \Rightarrow a=\frac{8}{3} \Rightarrow a^2=\frac{64}{9}$

From (1), $b^2=2\left(\frac{8}{3}\right)=\frac{16}{3} \Rightarrow b^2=\frac{16}{3}$

Equation of the ellipse is $\frac{9(x-2)^2}{64}+\frac{3(y+1)^2}{16}=1 \Rightarrow 9(x-2)^2+12(y+1)^2=64$

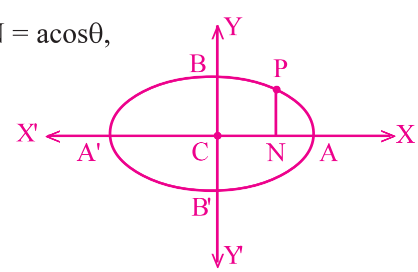
---

7 **C is the centre, AA' and BB' are major and minor axes of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. If (PN) is the ordinate of a point P on the ellipse then show that $\frac{(PN)^2}{(A'N)(AN)}=\frac{(BC)^2}{(CA)^2}$**

Sol: The parametric point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is P(θ) = (acosθ, bsinθ)

Now, y-coordinate PN = bsinθ and x-coordinate CN = acosθ,

Also CA = CA' = a and CB = CB' = b.



$$L.H.S=\frac{(PN)^2}{(A'N)(AN)}=\frac{(PN)^2}{(CA'+CN)(CA-CN)}$$

$$=\frac{(b\sin\theta)^2}{(a+a\cos\theta)(a-a\cos\theta)}=\frac{b^2\sin^2\theta}{a^2(1+\cos\theta)(1-\cos\theta)}$$

$$=\frac{b^2\sin^2\theta}{a^2(1-\cos^2\theta)}=\frac{b^2\sin^2\theta}{a^2\sin^2\theta}=\frac{b^2}{a^2}=\frac{(BC)^2}{(CA)^2}=R.H.S$$

8 **Find the equations to the tangents to the ellipse $x^2+2y^2=3$ drawn from the point (1,2) and also find the angle between these tangents.**

Sol: The equation of any line through (1,2) with slope m is (y−2)=m(x−1)

If y=mx+(2−m) is a tangent to the given ellipse $\frac{x^2}{3}+\frac{y^2}{3/2}=1$ then $c^2=a^2m^2+b^2$

$$\Rightarrow (2-m)^2 = 3m^2+\frac{3}{2} \Rightarrow 4m^2+8m-5=0 \Rightarrow (2m+5)(2m-1)=0 \Rightarrow m=\frac{1}{2} \text{ or } -\frac{5}{2}$$

∴ equations of the tangets are $(y-2)=\frac{1}{2}(x-1) \Rightarrow x-2y+3=0$

and $(y-2)=-\frac{5}{2}(x-1) \Rightarrow 5x+2y-9=0$

Acute angle between the above tangents is $\text{Tan}^{-1}\left|\frac{\frac{1}{2}+\frac{5}{2}}{1-\frac{1}{2}\cdot\frac{5}{2}}\right| = \text{Tan}^{-1}(12)$

---

9 **Show that the foot of the pernepdicular drawn from the centre on any tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ lies on the curve $(x^2+y^2)^2=a^2x^2+b^2y^2$.**

Sol: The equation of the tangent with slope m to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $y=mx+\sqrt{a^2m^2+b^2}$ ...(1)

The slope of its perpendicular is $-\frac{1}{m}$

The equation of the line passing through the centre (0,0) with slope $-\frac{1}{m}$ is $y=-\frac{1}{m}x$

Hence we have, $m=-\frac{x}{y}$

From (1), we have $(y-mx)^2=a^2m^2+b^2$

$$\Rightarrow \left(y+\frac{x}{y}x\right)=a^2\frac{x^2}{y^2}+b^2 \Rightarrow (y^2+x^2)^2=a^2x^2+b^2y^2$$

∴ The locus of $P(x_1,y_1)$ is $x^2+y^2=a^2$, which is the auxiliary circle.

10 **A man running on a race notices that the sum of the distances of the two flag posts from him is always 10m and the distance between the flag posts is 8m. Find the equation of the race course traced by the man.**

Sol: We compare the given situation with an ellipse in the standard form.

Here, we take the flag posts as the foci S,S' of the ellipse and the man as the point P on the ellipse.

We know that SP+S'P = 2a

Given that SP+S'P =10 = 2(5) $\Rightarrow$ a = 5

The distance between Flag posts = The distance between foci.

$\Rightarrow 2ae = 8 \Rightarrow ae = 4 \Rightarrow 5e = 4 \Rightarrow e = 4/5$

Now , $b^2 = a^2(1-e^2) = 5^2\left(1-\frac{16}{25}\right) = 25\left(\frac{9}{25}\right) = 9$

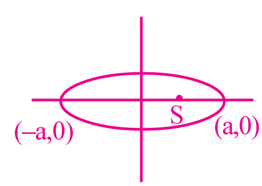
$\therefore$ The equation of the race course is given by $\frac{x^2}{25}+\frac{y^2}{9}=1$

---

11 **Show that among the points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (a > b), (–a, 0) is the farthest point and (a, 0) is the nearest point from the focus (ae, 0)**

Sol: Let P = (x, y) be any point on the ellipse so that $-a \le x \le a$ and S = (ae, 0) be the focus

P(x, y) is on the ellipse then $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



$\Rightarrow \frac{y^2}{b^2} = 1-\frac{x^2}{a^2} = \frac{a^2-x^2}{a^2} \Rightarrow y^2 = \frac{b^2}{a^2}(a^2-x^2)$

$\Rightarrow y^2 = (1-e^2)(a^2-x^2)$ ....(1) [$\because b^2 = a^2(1-e^2)$]

Then $SP^2 = (x-ae)^2 + (y-0)^2 = (x-ae)^2 + (1-e)^2(a^2-x^2)$

$= -2xae + a^2 + e^2x^2 = [a-ex]^2$

$\Rightarrow SP^2 = [a-ex]^2 \Rightarrow SP = |a-ex|$

We have $-a \le x \le a \Rightarrow -ae \le xe \le ae \Rightarrow -ae-a \le xe-a \le ae-a$ ......(2)

$\therefore ex - a < 0 \Rightarrow SP = a - ex$ ....(3)

From (2) and (3), $ae + a \ge SP \ge a - ae \Rightarrow a - ae \le SP \le ae + a$

$\therefore$Max SP = a + ae when P = (–a, 0) and Min SP = a – ae when P = (a, 0)

Hence the nearest point is (a, 0) and the farthest one is (–a, 0).

12 **The Orbit of the Earth is an ellipse with eccentricity 1/60 with the Sun at one of its foci, the major axis being approximately 186×10^6 miles in length. Find the shortest and longest distance of the Earth from the Sun.**

Sol: Let us take the equation of the orbit of the Earth as $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, (a>b)

Given that the length of the major axis is 186×10^6 miles $\Rightarrow 2a = 186\times10^6 \Rightarrow a = 93\times10^6$

Also the eccentricity of the orbit is e = 1/60

The longest distance of the Earth from the Sun is $a+ae = 93\times10^6 + 93\times10^6\times\frac{1}{60}$

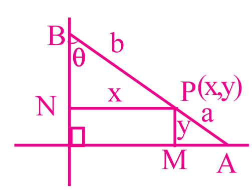
$$=93\times10^6\left(1+\frac{1}{60}\right)=93\times10^6\left(\frac{61}{60}\right)=9455\times10^4\text{ miles}$$

The shortest distance of the Earth from the Sun is $a-ae = 93\times10^6 - 93\times10^6\times\frac{1}{60}$

$$=93\times10^6\times\left(1-\frac{1}{60}\right)=93\times10^6\times\left(\frac{59}{60}\right)=9145\times10^4\text{ miles}$$

---

13 **A line of fixed length (a + b) moves so that its ends are always on two perpendicular straight lines fixed. Prove that a marked point on the line, which divides this line into portions of lengths 'a' and 'b' describes an ellipse and also find the eccentricity of the ellipse when a = 8, b = 12.**



Sol: Let the given line segment be taken as AB.

Suppose P(x, y) is any point on this line

such that AP = a, PB = b and AB=a+b.

Draw PM, PN perpendicular to x and y axis respectively

Let $\angle NBP = \theta$

From Δ PNB, $\sin\theta = \frac{PN}{PB}=\frac{x}{b}$ and from Δ PMA, $\sin(90^\circ-\theta) = \frac{MP}{PA}=\frac{y}{a}$

Squaring and adding we get $1=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ which is an ellipse.

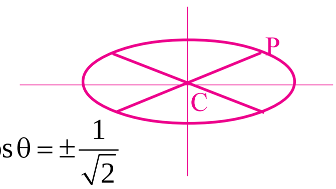
When a=, b=12, Eccentricity = $\sqrt{\frac{b^2-a^2}{b^2}}=\sqrt{\frac{144-64}{144}}=\sqrt{\frac{80}{144}}=\sqrt{\frac{5}{9}}=\frac{\sqrt{5}}{3}$

14 **The distance of a point on the ellipse $x^2+3y^2=6$ from its centre is equal to 2. Find the eccentric angles.**

Sol: Given ellipse is $x^2+3y^2=6 \Rightarrow \frac{x^2}{6}+\frac{y^2}{2}=1 \Rightarrow a^2=6 \Rightarrow a=\sqrt{6};\ b^2=2 \Rightarrow b=\sqrt{2}$

The parametric point on the ellipse is P(acosθ, bsinθ) = $P(\sqrt{6}\cos\theta, \sqrt{2}\sin\theta)$

Also, centre C = (0,0)



Now $CP=2 \Rightarrow CP^2=4 \Rightarrow 6\cos^2\theta+2\sin^2\theta=4$

$\Rightarrow 6\cos^2\theta+2(1-\cos^2\theta)=4 \Rightarrow 4\cos^2\theta=2 \Rightarrow \cos^2\theta=\frac{1}{2} \Rightarrow \cos\theta=\pm\frac{1}{\sqrt{2}}$

$\cos\theta=\frac{1}{\sqrt{2}} \Rightarrow \theta=\frac{\pi}{4},\frac{7\pi}{4};\ \cos\theta=-\frac{1}{\sqrt{2}} \Rightarrow \theta=\frac{3\pi}{4},\frac{5\pi}{4}$

∴ The eccentric angles are $\theta=\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}$

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15 **Prove that the equation of the chord joining the point α and β on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$**

Sol: Given points on the ellipse are P(a cos α, b sin α), Q (a cos β, b sin β).

Slope of $\overleftrightarrow{PQ}$ is $\frac{b\sin\alpha-b\sin\beta}{a\cos\alpha-a\cos\beta}=\frac{b(\sin\alpha-\sin\beta)}{a(\cos\alpha-\cos\beta)}$

Equation of $\overleftrightarrow{PQ}$ is $y-b\sin\alpha=\frac{b(\sin\alpha-\sin\beta)}{a(\cos\alpha-\cos\beta)}(x-a\cos\alpha)$

$\Rightarrow a(y-b\sin\alpha)(\cos\alpha-\cos\beta)=b(\sin\alpha-\sin\beta)(x-a\cos\alpha)$

$\Rightarrow \frac{x-a\cos\alpha}{a}(\sin\alpha-\sin\beta)=\frac{y-b\sin\alpha}{b}(\cos\alpha-\cos\beta)$

$\Rightarrow \left(\frac{x}{a}-\cos\alpha\right)\cancel{2}\cos\frac{\alpha+\beta}{2}\cancel{\sin\frac{\alpha-\beta}{2}}=\left(\frac{y}{b}-\sin\alpha\right)(\cancel{-2})\sin\frac{\alpha+\beta}{2}\cancel{\sin\frac{\alpha-\beta}{2}}$

$\Rightarrow \left(\frac{x}{a}-\cos\alpha\right)\cos\frac{\alpha+\beta}{2}=-\left(\frac{y}{b}-\sin\alpha\right)\sin\frac{\alpha+\beta}{2}$

$\Rightarrow \frac{x}{a}\left(\cos\frac{\alpha+\beta}{2}\right)-\cos\alpha\cos\left(\frac{\alpha+\beta}{2}\right)=-\frac{y}{b}\left(\sin\frac{\alpha+\beta}{2}\right)+\sin\alpha\sin\left(\frac{\alpha+\beta}{2}\right)$

$\Rightarrow \frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\alpha\cos\frac{\alpha+\beta}{2}+\sin\alpha\sin\frac{\alpha+\beta}{2}=\cos\left(\alpha-\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$

$\therefore \frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$

16 **If θ_1 , θ_2 are the eccentric angles of the extremeties of a focal chord (other that the vertices) of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (a > b) and e its eccentricity. Then show that**

i) $e\cos\frac{(\theta_1+\theta_2)}{2}=\cos\frac{(\theta_1-\theta_2)}{2}$ ii) $\frac{e+1}{e-1}=\cot\left(\frac{\theta_1}{2}\right).\cot\left(\frac{\theta_2}{2}\right)$ EAM Q

Sol: Equation of the ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,(a>b)$

Let P (θ_1), Q (θ_2) are the ends of the focal chord $\Rightarrow$ Slope of $\overleftrightarrow{PS}$ = Slope of $\overleftrightarrow{QS}$

$$\Rightarrow \frac{\not{b}\sin\theta_1}{\not{a}(\cos\theta_1-e)}=\frac{\not{b}\sin\theta_2}{\not{a}(\cos\theta_2-e)}\Rightarrow \sin\theta_1(\cos\theta_2-e)=\sin\theta_2(\cos\theta_1-e)$$

$$\Rightarrow \sin\theta_1.\cos\theta_2 - e\sin\theta_1 = \cos\theta_1\sin\theta_2 - e\sin\theta_2$$

$$\Rightarrow \sin\theta_1.\cos\theta_2 - \cos\theta_1\sin\theta_2 = e\sin\theta_1 - e\sin\theta_2$$

$$\Rightarrow \sin(\theta_1-\theta_2) = e(\sin\theta_1 - \sin\theta_2)$$

$$\Rightarrow 2\sin\frac{(\theta_1-\theta_2)}{2}.\cos\frac{(\theta_1-\theta_2)}{2}=e\left[2\cos\frac{\theta_1+\theta_2}{2}.\sin\frac{\theta_1-\theta_2}{2}\right]$$

$$\Rightarrow e\cos\left(\frac{\theta_1+\theta_2}{2}\right)=\cos\left[\frac{\theta_1-\theta_2}{2}\right] \text{------------}(1)$$

From (1), $\frac{e}{1}=\frac{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}{\cos\left(\frac{\theta_1+\theta_2}{2}\right)}$

Applying componendo and dividendo rule, we have

$$\frac{e+1}{e-1}=\frac{\cos\left(\frac{\theta_1-\theta_2}{2}\right)+\cos\left(\frac{\theta_1+\theta_2}{2}\right)}{\cos\left(\frac{\theta_1-\theta_2}{2}\right)-\cos\left(\frac{\theta_1+\theta_2}{2}\right)}=\frac{2\cos\frac{\theta_1}{2}.\cos\frac{\theta_2}{2}}{2\sin\frac{\theta_1}{2}.\sin\frac{\theta_2}{2}}=\cot\left(\frac{\theta_1}{2}\right).\cot\left(\frac{\theta_2}{2}\right)$$