WELCOME

STAR 'QR CODE'

DIGITAL MATERIAL

ADDITION OF VECTORS -INDEX

4.ADDITION OF VECTORS

1. INTRODUCTION PAGE

Sections	No. of periods (15 to 18)	Weightage in IPE [2x2]+[1x4]=8
1. Fundamental concepts and addition of vectors	8	2 Marks
2. Linear Combination of Vectors, Coplanar and non-coplanar vectors.	4	4 Marks
3. Vector equations of lines and planes in parametric forms.	4	2 Marks

The concepts of vectors and their applications are widely used in various branches of Science and Engineering like Physics, Aero Engineering, Space Technology. Some concepts of Vector Algebra and Geometry reflect each other (i.e., same thoughts are expressed in different languages).

In the First section, the fundamental concepts like representation of vectors, scalar multiplication of vectors, angle between vectors, Types of vectors and section formulae are discussed."The triangle law" of addition of vectors explains the process of adding 2 given vectors. The structural properties of vectors w.r.t addition of vectors are discussed. Triangular inequality is stated and proved. Various geometrical problems are dealt within the vector mode.

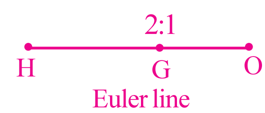
The concept of linear combination *of elements is an important concept of Modern Algebra. Here, in section-2, the linear combination of vectors is introduced. Two important concepts, Collinearity of three points (2 vectors) and coplanarity of 4 points (3 vectors) are discussed interms of linear dependence and linear independence of vectors.*

In Section 3, the vector equations of lines and planes in parametric form are stated and proved. Some simple applications on these models are included.

2. ADDITIONAL QS ON VECTOR ADDITION

1. In ΔABC if O is the circumcentre and H is the orthocentre then show that

(i) $\overline{HA}+\overline{HB}+\overline{HC}=2\overline{HO}$ (ii) $\overline{OA}+\overline{OB}+\overline{OC}=\overline{OH}$



Sol: Let us take the orthocentre H as the origin of reference

$\Rightarrow$ the P.Vs of A,B,C,O,H are $\overline{HA},\overline{HB},\overline{HC},\overline{HO}$ and $\overline{HH}=\overline{0}$

If G is the centroid of ΔABC then $\overline{HG}=\dfrac{\overline{HA}+\overline{HB}+\overline{HC}}{3}$ ..... (1)

Also, we know that the centroid G divides $\overline{HO}$ in the ratio 2:1

$\Rightarrow \overline{HG}=\dfrac{2\overline{HO}+\overline{HH}}{2+1}=\dfrac{2\overline{HO}+\overline{0}}{3}=\dfrac{2\overline{HO}}{3}$ .... (2)

From (1), (2), $\dfrac{\overline{HA}+\overline{HB}+\overline{HC}}{3}=\dfrac{2\overline{HO}}{3}\Rightarrow \overline{HA}+\overline{HB}+\overline{HC}=2\overline{HO}$

Hence (i) is proved.

Now, consider $\overline{HA}+\overline{HB}+\overline{HC}=2\overline{HO}$

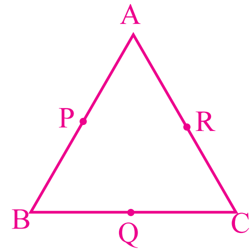
$\Rightarrow(\overline{HO}+\overline{OA})+(\overline{HO}+\overline{OB})+(\overline{HO}+\overline{OC})=2\overline{HO}\Rightarrow\overline{OA}+\overline{OB}+\overline{OC}+3\overline{HO}=2\overline{HO}$

$\Rightarrow\overline{OA}+\overline{OB}+\overline{OC}=2\overline{HO}-3\overline{HO}=-\overline{HO}=\overline{OH}$

Hence (ii) is proved.

2. In ΔABC, P,Q and R are the mid poins of the sides AB, BC and CA respectively. If D is any point (i) then express $\overline{DA}+\overline{DB}+\overline{DC}$ interms of $\overline{DP}$, $\overline{DQ}$ and $\overline{DR}$

(ii) If $\overline{PA}+\overline{QB}+\overline{RC}=\overline{\alpha}$, then find $\overline{\alpha}$.



Sol: (i) We take D as the origin of reference.

P is the mid point of AB $\Rightarrow \overline{DP}=\dfrac{\overline{DA}+\overline{DB}}{2} \Rightarrow \overline{DA}+\overline{DB}=2\overline{DP}$......(1)

Q is the mid point of BC $\Rightarrow \overline{DQ}=\dfrac{\overline{DB}+\overline{DC}}{2} \Rightarrow \overline{DB}+\overline{DC}=2\overline{DQ}$......(2)

R is the mid point of CA $\Rightarrow \overline{DR}=\dfrac{\overline{DC}+\overline{DA}}{2} \Rightarrow \overline{DC}+\overline{DA}=2\overline{DR}$......(3)

$(1)+(2)+(3) \Rightarrow 2(\overline{DA}+\overline{DB}+\overline{DC})=2(\overline{DP}+\overline{DQ}+\overline{DR}) \Rightarrow \overline{DA}+\overline{DB}+\overline{DC}=\overline{DP}+\overline{DQ}+\overline{DR}$

(ii) Given that $\overline{PA}+\overline{QB}+\overline{RC}=\overline{\alpha}$

Now $\overline{PA}+\overline{QB}+\overline{RC}=(\overline{DA}-\overline{DP})+(\overline{DB}-\overline{DQ})+(\overline{DC}-\overline{DR})$

$=(\overline{DA}+\overline{DB}+\overline{DC})-(\overline{DP}+\overline{DQ}+\overline{DR})=\overline{0}$ (from (i)) $\therefore \overline{\alpha}=\overline{0}$

---

3. If $\overline{a}+\overline{b}+\overline{c}=\alpha\overline{d}$, $\overline{b}+\overline{c}+\overline{d}=\beta\overline{a}$ and $\overline{a},\overline{b},\overline{c}$ are non-coplanar vectors then show that $\overline{a}+\overline{b}+\overline{c}+\overline{d}=\overline{0}$.

Sol: Given that $\overline{a}+\overline{b}+\overline{c}=\alpha\overline{d}$ ....(1) Also, $\overline{b}+\overline{c}+\overline{d}=\beta\overline{a} \Rightarrow \overline{d}=\beta\overline{a}-\overline{b}-\overline{c}$

Now (1) $\Rightarrow \overline{a}+\overline{b}+\overline{c}=\alpha(\beta\overline{a}-\overline{b}-\overline{c})=\alpha\beta\overline{a}-\alpha\overline{b}-\alpha\overline{c}$

$\Rightarrow (1-\alpha\beta)\overline{a}+(1+\alpha)\overline{b}+(1+\alpha)\overline{c}=\overline{0}$

Given that $\overline{a},\overline{b},\overline{c}$ are non-coplanar vectors.Hence, $1-\alpha\beta=0$ and $1+\alpha=0$

$\Rightarrow \alpha=-1$ and $\alpha\beta=1 \Rightarrow (-1)\beta=1 \Rightarrow \beta=-1$

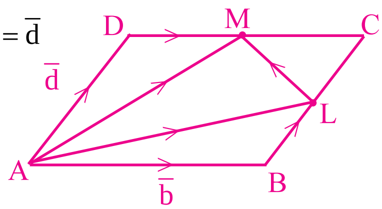
$\therefore (1) \Rightarrow \overline{a}+\overline{b}+\overline{c}=(-1)\overline{d} \Rightarrow \overline{a}+\overline{b}+\overline{c}+\overline{d}=\overline{0}$

4. ABCD is a parallelogram. If L and M are the middle points of BC and CD respectively then find (i) $\overline{AL}$ and $\overline{AM}$ interms of $\overline{AB}$ and $\overline{AD}$ ii) λ, If $\overline{AM} = \lambda\,\overline{AD} - \overline{LM}$

EAM Q

Sol: ABCD is a parallelogram $\Rightarrow \overline{AB} = \overline{DC}, \overline{AD} = \overline{BC}$

Let A be the origin of reference so that $\overline{AB} = \bar{b}$ and $\overline{AD} = \bar{d}$



L is the midpoint of the side $\overline{BC} \Rightarrow \overline{BL} = \overline{LC} = \frac{1}{2}\overline{BC}$

M is the midpoint of the side $\overline{DC} \Rightarrow \overline{CM} = \overline{MD} = \frac{1}{2}\overline{CD}$

i) $\overline{AL} = \overline{AB} + \overline{BL} = \overline{AB} + \frac{1}{2}\overline{BC} = \overline{AB} + \frac{1}{2}\overline{AD} = \bar{b} + \frac{\bar{d}}{2}$ ....(1)

Now, $\overline{AM} = \overline{AD} + \overline{DM} = \overline{AD} + \frac{1}{2}\overline{DC} = \overline{AD} + \frac{1}{2}\overline{AB} = \bar{d} + \frac{\bar{b}}{2}$ ....(2)

ii) From ΔALM, $\overline{AL} + \overline{LM} = \overline{AM} \Rightarrow \overline{LM} = \overline{AM} - \overline{AL}$

$$= \left(\bar{d} + \frac{\bar{b}}{2}\right) - \left(\bar{b} + \frac{\bar{d}}{2}\right) = \frac{\bar{d}}{2} - \frac{\bar{b}}{2} = \frac{\bar{d} - \bar{b}}{2}$$

Given that $\overline{AM} = \lambda\overline{AD} - \overline{LM}$

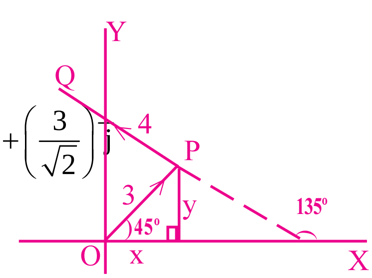
$$\Rightarrow \bar{d} + \frac{\bar{b}}{2} = \lambda(\bar{d}) - \left(\frac{\bar{d} - \bar{b}}{2}\right) \Rightarrow \bar{d} + \frac{\bar{b}}{2} = \lambda\bar{d} - \frac{\bar{d}}{2} + \frac{\bar{b}}{2} \Rightarrow \lambda\bar{d} = \bar{d} + \frac{\bar{d}}{2} = \frac{3}{2}\bar{d} \Rightarrow \lambda = \frac{3}{2}$$

---

5. In a cartesian plane, O is the origin of the coordinate axes. A person starts at O and walks a distance of 3 units in the NORTH-EAST direction and reaches the point P. From P he walks 4 units distance parallel to NORTH-WEST direction and reaches the point Q. Express the vector $\overline{OQ}$ in terms of $\bar{i}$ and $\bar{j}$ (observe that $\underline{|XOP} = 45^\circ$)

Sol: The person starts at O and walks a distance of 3 units in the NORTH -EAST direction

$\Rightarrow |\overline{OP}| = 3\text{units}$ and $\underline{|XOP} = 45^\circ$



$\therefore \overline{OP} = (3\cos 45^\circ)\bar{i} + (3\sin 45^\circ)\bar{j}$ $\quad \Rightarrow \overline{OP} = \left(\frac{3}{\sqrt{2}}\right)\bar{i} + \left(\frac{3}{\sqrt{2}}\right)\bar{j}$

Also $|\overline{PQ}| = 4\text{units}$

$\overline{PQ}$ makes 135° with X-axis.

$$\therefore \overline{PQ} = 4\cos(135^\circ)\bar{i} + 4\sin(135^\circ)\bar{j} = \frac{-4}{\sqrt{2}}\bar{i} + \frac{4}{\sqrt{2}}\bar{j}$$

$$\therefore \overline{OQ} = \overline{OP} + \overline{PQ} = \left[\left(\frac{3}{\sqrt{2}}\right)\bar{i} + \left(\frac{3}{\sqrt{2}}\right)\bar{j}\right] + \left[\frac{-4}{\sqrt{2}}\bar{i} + \frac{4}{\sqrt{2}}\bar{j}\right] = \frac{-\bar{i} + 7\bar{j}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left(-\bar{i} + 7\bar{j}\right)$$

6. The points O, A, B, X and Y are such that $\overline{OA} = \bar{a}, \overline{OB} = \bar{b}, \overline{OX} = 3\bar{a}$ and $\overline{OY} = 3\bar{b}$. Find $\overline{BX}$ and $\overline{AY}$ interms of $\bar{a}$ and $\bar{b}$. Further, if the point P divides $\overline{AY}$ in the ratio 1:3. Express $\overline{BP}$ interms of $\bar{a}$ and $\bar{b}$

Sol: Given $\overline{OA} = \bar{a},\ \overline{OB} = \bar{b}, \overline{OX} = 3\bar{a}, \overline{OY} = 3\bar{b}$

$\overline{BX} = \overline{OX} - \overline{OB} = 3\bar{a} - \bar{b}$; $\qquad \overline{AY} = \overline{OY} - \overline{OA} = 3\bar{b} - \bar{a}$

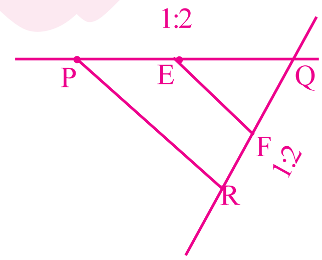
P divides $\overline{AY}$ in the ratio 1:3 $\therefore$ Position vector of P is $\overline{OP} = \dfrac{1(\overline{OY}) + 3(\overline{OA})}{1+3} = \dfrac{3\bar{b} + 3\bar{a}}{4}$

Now, $\overline{BP} = \overline{OP} - \overline{OB} = \left(\dfrac{3\bar{b} + 3\bar{a}}{4}\right) - \bar{b} = \dfrac{3\bar{b} + 3\bar{a} - 4\bar{b}}{4} = \dfrac{3\bar{a} - \bar{b}}{4}$

---

7. The point E divides the segment PQ internally in the ratio 1:2 and R is any point not on the line PQ. If F is a point on QR such that QF:FR=2:1. Show that EF is parallel to PR.

Sol: Let 'O' be the origin and $\overline{OP} = \bar{p}, \overline{OQ} = \bar{q}$ and $\overline{OR} = \bar{r}$



E divides PQ in the ratio 1:2

$\therefore$ Position Vector of E is $\overline{OE} = \dfrac{1(\bar{q}) + 2(\bar{p})}{1+2} = \dfrac{2\bar{p} + \bar{q}}{3}$

Also, F is a point on QR such that QF:FR=2:1

$\therefore$ Position Vector of F is $\overline{OF} = \dfrac{2(\bar{r}) + 1(\bar{q})}{2+1} = \dfrac{\bar{q} + 2\bar{r}}{3}$

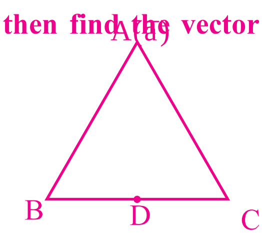
To claim that EF||PR we have to show that $\overline{EF} = \lambda\, \overline{PR}$, $\lambda \in R$

Now, $\overline{EF} = \overline{OF} - \overline{OE} = \left(\dfrac{\bar{q} + 2\bar{r}}{3}\right) - \left(\dfrac{2\bar{p} + \bar{q}}{3}\right) = \dfrac{2}{3}(\bar{r} - \bar{p}) = \dfrac{2}{3}(\overline{OR} - \overline{OP}) = \dfrac{2}{3}\overline{PR}$

$\therefore \overline{EF} = \dfrac{2}{3}\overline{PR}$ Hence, $\overline{EF}$ is parallel to $\overline{PR}$

---

8. If $\bar{a}, \bar{b}, \bar{c}$ are P.V's of the vertices A,B,C respectively of ΔABC then find the vector equation of the median through the vertex A.



Sol: Let O be the origin of reference so that $\overline{OA} = \bar{a}, \overline{OB} = \bar{b}, \overline{OC} = \bar{c}$

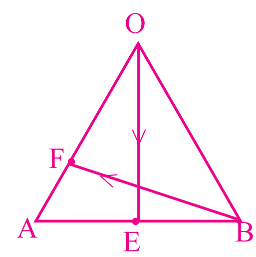
Let D be the mid point of $\overline{BC} \Rightarrow \overline{OD} = \dfrac{\overline{OB} + \overline{OC}}{2} = \dfrac{1}{2}(\bar{b} + \bar{c})$

$\therefore$ the vector equation of the median through $A(\bar{a})$ and $D\left(\dfrac{1}{2}(\bar{b} + \bar{c})\right)$ is

$\bar{r} = (1-t)\overline{OA} + t(\overline{OD}), t \in R \Rightarrow \bar{r} = (1-t)\bar{a} + t\dfrac{1}{2}(\bar{b} + \bar{c}), t \in R$

9. In ΔOAB, E is the midpoint of AB and F is a point on OA such that $\overline{OF} = 2\overline{FA}$. If C is the point of intersection of OE and BF, then find the ratio's OC:CE and BC:CF

Sol: Let 'O' be the origin of reference and $\overline{OA} = \bar{a}, \overline{OB} = \bar{b}$



E is the midpoint of AB $\Rightarrow \overline{OE} = \frac{1}{2}(\bar{a} + \bar{b})$

Also F is a point on OA such that $\overline{OF} = 2\ \overline{FA}$

$\Rightarrow$F divides OA in the ratio 2:1

$\therefore \overline{OF} = \frac{2(\bar{a}) + 1(o)}{2+1} = \frac{2}{3}\bar{a}$

C is the point of intersection of OE and BF.

Let C divides OE in the ratio 1:λ

$\Rightarrow \overline{OC} = \frac{1(\overline{OE}) + \lambda(\bar{0})}{1+\lambda} = \frac{\frac{1}{2}(\bar{a}+\bar{b})}{\lambda+1}$..........(1)

Let C divides BF in the ratio μ:1

$\Rightarrow \overline{OC} = \frac{\mu(\overline{OF}) + 1(\overline{OB})}{\mu+1} \Rightarrow \overline{OC} = \frac{\mu\left(\frac{2}{3}\bar{a}\right) + 1(\bar{b})}{\mu+1}$..........(2)

From (1) & (2), $\frac{1}{2}\frac{(\bar{a}+\bar{b})}{\lambda+1} = \frac{\frac{2}{3}\mu(\bar{a}) + (\bar{b})}{\mu+1}$

Equating the coefficients of $\bar{a}$ we get $\frac{1}{2(\lambda+1)} = \frac{2}{3}.\frac{\mu}{\mu+1}$........(3)

Equating the coefficients of $\bar{b}$ we get $\frac{1}{2(\lambda+1)} = \frac{1}{\mu+1}$........(4)

From (3), $\frac{2}{3}.\frac{\mu}{\cancel{\mu+1}} = \frac{1}{\cancel{\mu+1}} \Rightarrow \frac{2}{3}\mu = 1 \Rightarrow \mu = \frac{3}{2}$

From (4), $\frac{1}{2(\lambda+1)} = \frac{1}{\left(\frac{3}{2}+1\right)} \Rightarrow \frac{1}{2(\lambda+1)} = \frac{2}{5} \Rightarrow 4(\lambda+1) = 5 \Rightarrow \lambda+1 = \frac{5}{4} \Rightarrow \lambda = \frac{1}{4}$

$\therefore$ C divides OE in the ratio $1:\lambda = 1:\frac{1}{4} = 4:1$ $\therefore$ OC:CE=4:1

Also C divides BF in the ratio $\mu:1 = \frac{3}{2}:1 = 3:2$ $\therefore$ BC:CF=3:2

10. ABCD is a trapezium in which AB and CD are parallel. Prove by vector method that the mid points of the sides AB, CD and the intersection of the diagonals are collinear.

Sol: ABCD is a trapezium $\Rightarrow$AB is parallel to DC

Let $A(\bar{0})$ be the origin so that $\overline{AB}=\bar{b}$, $\overline{AC}=\bar{c}$ and $\overline{AD}=\bar{d}$

Now $\overline{DC}=\overline{AC}-\overline{AD}=\bar{c}-\bar{d}$ But $\overline{AB}\,||\,\overline{DC}\Rightarrow \bar{b}=\lambda(\bar{c}-\bar{d})$.........(1) for some $\lambda\neq0,\lambda\in R$

Equation of the diagonal $\overline{AC}$ is $\bar{r}=(1-t)\bar{0}+t\bar{c}$, where t is scalar $\Rightarrow \bar{r}=t\bar{c}$ ....(2)

Equation of the diagonal $\overline{BD}$ is $\bar{r}=(1-s)\bar{b}+s\bar{d}$,......(3) for some scalar $s\in R$

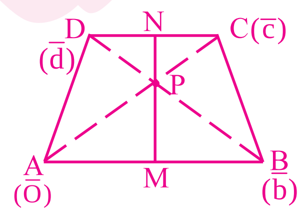
Let P be the point of intersection of the diagonals AC and BD and its P.V is $\overline{AP}=\bar{r}$

From (2) and (3), $t\bar{c}=(1-s)\bar{b}+s\bar{d}$

$\Rightarrow t\bar{c}=(1-s)\lambda(\bar{c}-\bar{d})+s\bar{d}$, from (1) $\Rightarrow t\bar{c}=(1-s)\lambda\bar{c}-[\lambda(1-s)-s]\bar{d}$

Equating the coefficients of $\bar{c}$ and $\bar{d}$ on both sides we get

$t=(1-s)\lambda$ and $\lambda(1-s)-s=0\Rightarrow s(1+\lambda)=\lambda\Rightarrow s=\dfrac{\lambda}{1+\lambda}$



Now, $t=(1-s)\lambda=\left(1-\dfrac{\lambda}{\lambda+1}\right)\lambda=\left(\dfrac{\lambda+1-\lambda}{\lambda+1}\right)\lambda=\dfrac{\lambda}{\lambda+1}$

From (2), $\overline{AP}=\bar{r}=\left(\dfrac{\lambda}{\lambda+1}\right)\bar{c}$

To claim that M,P,N are collinear we show that $\overline{PM}=\lambda(\overline{NP})$

M is the midpoint of AB $\Rightarrow \overline{AM}=\dfrac{1}{2}\bar{b}$,

N is the mid point of $\overline{DC}\Rightarrow\overline{AN}=\dfrac{1}{2}(\bar{c}+\bar{d})$ $\left[\because \overline{AN}=\dfrac{\overline{AC}+\overline{AD}}{2}\right]$

Now $\overline{PM}=\overline{AM}-\overline{AP}=\dfrac{1}{2}\bar{b}-\left[\dfrac{\lambda}{\lambda+1}\right]\bar{c}=\dfrac{1}{2}\lambda(\bar{c}-\bar{d})-\dfrac{\lambda}{\lambda+1}\bar{c}$, from (1)

$$=\left[\frac{\lambda}{2}-\frac{\lambda}{\lambda+1}\right]\bar{c}-\frac{\lambda}{2}\bar{d}=\left[\frac{\lambda^2+\lambda-2\lambda}{2(\lambda+1)}\right]\bar{c}-\frac{\lambda}{2}\bar{d}=\left[\frac{\lambda^2-\lambda}{2(\lambda+1)}\right]\bar{c}-\frac{\lambda}{2}\bar{d}=\lambda\left[\left(\frac{\lambda-1}{\lambda+1}\right)\bar{c}-\frac{\bar{d}}{2}\right]\text{........(4)}$$

Now $\overline{NP}=\overline{AP}-\overline{AN}=\left(\dfrac{\lambda}{\lambda+1}\right)\bar{c}-\dfrac{1}{2}(\bar{c}+\bar{d})=\left(\dfrac{\lambda}{\lambda+1}-\dfrac{1}{2}\right)\bar{c}-\dfrac{1}{2}\bar{d}$

$$=\left(\frac{2\lambda-\lambda-1}{\lambda+1}\right)\bar{c}-\frac{1}{2}\bar{d}=\left(\frac{\lambda-1}{\lambda+1}\right)\bar{c}-\frac{1}{2}\bar{d}\text{.........(5)}$$

From (4) and (5), $\overline{PM}=\lambda(\overline{NP})$, $\lambda\neq0,\lambda\in R$

$\therefore$ M,P,N are collinear

Hence the midpoints of parallel sides of a trapezium and the point of intersection of the diagonals are collinear.

3. PROOFS OF THEOREMS ON COLLINEAR, COPLANAR & NON-COPLANAR VECTORS

1. **Linear Combination of vectors**:

If $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n$ are n vectors and $x_1, x_2, \ldots, x_n$ are n scalars then the vector of the form $x_1\bar{a}_1 + x_2\bar{a}_2 + \ldots + x_n\bar{a}_n$ is called a linear combination of the given vectors.

2. Collinearity and non collinearity of 2 vectors (3 points)

Theorem 1.1.: If $\bar{a} = t\bar{b}$ for some scalar t then the vectors $\bar{a}, \bar{b}$ are collinear.

Proof: $t\bar{b}$ and $\bar{b}$ are collinear for any $t \in R \Rightarrow \bar{a}, \bar{b}$ are collinear.

Theorem 1.2: Two non-zero vectors $\bar{a}, \bar{b}$ are collinear $\Leftrightarrow x\bar{a} + y\bar{b} = \bar{0}$, for some scalars x,y not both zero.

Proof: **Part 1**: Suppose $\bar{a}, \bar{b}$ are collinear.

$\Rightarrow \bar{a} = t\bar{b}$ for some scalar $t \in R$

$\Rightarrow \bar{a} - t\bar{b} = \bar{0} \Rightarrow 1(\bar{a}) - t\bar{b} = \bar{0}$, which is in the form $x\bar{a} + y\bar{b} = \bar{0}$, here $x = 1 \neq 0$, $-t \in R$

Part 2: Conversely suppose that $x\bar{a} + y\bar{b} = \bar{0}$ for some scalars x,y not both zero.

Let $x \neq 0$, now $x\bar{a} + y\bar{b} = \bar{0} \Rightarrow x\bar{a} = -y\bar{b} \Rightarrow \bar{a} = -(y/x)\bar{b}$. $\Rightarrow \bar{a}, \bar{b}$ are collinear.

Theorem 1.3.: If $\bar{a}, \bar{b}$ are 2 non zero and non collinear vectors then $x\bar{a} + y\bar{b} = \bar{0} \Rightarrow x=0, y=0$.

Proof: Given that $\bar{a}, \bar{b}$ are non collinear vectors.

On the contrary suppose that $x \neq 0$ then $x\bar{a} + y\bar{b} = \bar{0} \Rightarrow x\bar{a} = -y\bar{b} \Rightarrow \bar{a} = (-y/x)\bar{b}$

$\Rightarrow \bar{a}, \bar{b}$ are collinear.

This is a contradiction

Hence the supposition that $x \neq 0$ is false $\Rightarrow x=0$

Similarly, it can be shown that y=0

Theorem on the Condition for collinearity of 3 points

Theorem 2: Three points A,B,C with position vectors $\bar{a}, \bar{b}, \bar{c}$ are collinear $\Leftrightarrow x\bar{a} + y\bar{b} + z\bar{c} = \bar{0}$ for some scalars x,y,z not all zero and x+y+z=0.

Proof: **Part 1**: Suppose the 3 points A,B,C are collinear

$\Rightarrow$ the vectors $\overline{AB}, \overline{AC}$ are collinear $\Rightarrow \overline{AB} = t\,\overline{AC}$ for some scalar $t \in R$

$\Rightarrow \overline{OB} - \overline{OA} = t(\overline{OC} - \overline{OA}) \Rightarrow \bar{b} - \bar{a} = t(\bar{c} - \bar{a}) \Rightarrow \bar{b} - \bar{a} = t\bar{c} - t\bar{a}$

$\Rightarrow t\bar{a} - \bar{a} + \bar{b} - t\bar{c} = \bar{0} \Rightarrow (t-1)\bar{a} + 1(\bar{b}) + (-t)\bar{c} = \bar{0}$ which is in the form $x\bar{a} + y\bar{b} + z\bar{c} = \bar{0}$

for $y = 1 \neq 0$

comparing with $x\bar{a} + y\bar{b} + z\bar{c} = 0$ we get x=t−1, y=1, z=−t.

$\therefore$ x+y+z=t−1+1−t=0.

Part 2: Conversely, suppose that $x\bar{a} + y\bar{b} + z\bar{c} = \bar{0}$ for some scalars x,y,z not all zero & x+y+z=0

Let $x \neq 0$, now $x+y+z=0 \Rightarrow z = -(x+y)$

$\therefore x\bar{a} + y\bar{b} + z\bar{c} = \bar{0} \Rightarrow x\bar{a} + y\bar{b} - (x+y)\bar{c} = \bar{0}$

$\Rightarrow x\bar{a} + y\bar{b} - x\bar{c} - y\bar{c} = \bar{0} \Rightarrow x(\bar{a} - \bar{c}) + y(\bar{b} - \bar{c}) = \bar{0}$

$\Rightarrow x(\overline{OA} - \overline{OC}) + y(\overline{OB} - \overline{OC}) = \bar{0} \Rightarrow x(\overline{CA}) + y(\overline{CB}) = \bar{0}$

Here $x \neq 0$ $\therefore$ the two vectors $\overline{CA}, \overline{CB}$ are collinear vectors.

$\Rightarrow$ the three points A,B,C are collinear.

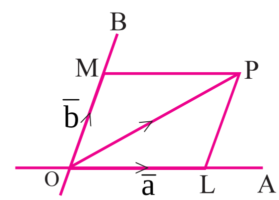
A Conclusion: Three points $A(\bar{a})$, $B(\bar{b})$, $C(\bar{c})$ are collinear the vectors

$\overline{AB}$ & $\overline{AC}$ or $\overline{BA}$ & $\overline{BC}$ or $\overline{CA}$ & $\overline{CB}$ are collinear.

Theorem on the resolution of a vector in a plane

Theorem 3: If $\bar{a}, \bar{b}$ are 2 non collinear vectors and $\bar{r}$ is any vector in the plane generated by $\bar{a}, \bar{b}$ then there exists a unique pair of reals x,y such that $\bar{r} = x\bar{a} + y\bar{b}$.

Proof: We prove that $\bar{r} = x\bar{a} + y\bar{b}$ for some scalars x and y and then we prove the uniqueness of x&y.



Let $\bar{a} = \overline{OA}, \bar{b} = \overline{OB}, \bar{r} = \overline{OP}$

Let the lines parallel to $\bar{b}, \bar{a}$ through P cuts OA and OB at L,M

$\Rightarrow$OLPM is a parallelogram $\Rightarrow \overline{OL} = \overline{MP}$ and $\overline{OM} = \overline{LP}$

Now, $\overline{OL}, \overline{OA}$ are collinear $\Rightarrow \overline{OL} = x\overline{OA}$ for some scalar x

Also $\overline{OM}, \overline{OB}$ are collinear $\Rightarrow \overline{OM} = y\overline{OB}$ for some scalar y.

Now, $\bar{r} = \overline{OP} = \overline{OL} + \overline{LP} = \overline{OL} + \overline{OM} = x\overline{OA} + y\overline{OB} = x\bar{a} + y\bar{b}$.

Uniqueness part:

For if $\exists$ another set of scalars s,t such that $\bar{r} = s\bar{a} + t\bar{b}$ then

we have $\bar{r} = x\bar{a} + y\bar{b} = s\bar{a} + t\bar{b}$

$\Rightarrow x\bar{a} - s\bar{a} + y\bar{b} - t\bar{b} = \bar{0}$

$\Rightarrow (x-s)\bar{a} + (y-t)\bar{b} = \bar{0}$

$\Rightarrow x-s=0$ and $y-t=0$ $\quad \because \bar{a}, \bar{b}$ are noncollinear $\Rightarrow \bar{a}, \bar{b}$ are linearly independent.

$\Rightarrow x=s$ and $y=t$

$\therefore \bar{r} = x\bar{a} + y\bar{b}$ for some unique pair of reals x and y

Result: If $\bar{a}, \bar{b}$ are any 2 non collinear vectors and $\bar{r}$ is any vector such that $\bar{r} = x\bar{a} + y\bar{b}$ for some scalars x,y then $\bar{r}$ is coplanar with $\bar{a}, \bar{b}$.

3. Coplanarity and non-coplanarity of 3 vectors (4 points)

Theorem 1.1: If $\bar{a}, \bar{b}$ are 2 non collinear vectors and $\bar{c}$ is any vector then $\bar{a}, \bar{b}, \bar{c}$ are coplanar $\Leftrightarrow x\bar{a} + y\bar{b} = \bar{c}$ for some scalars x,y.

Proof: Suppose $\bar{a}, \bar{b}, \bar{c}$ are coplanar.

Then $\bar{c}$ lies in the plane generated by $\bar{a}, \bar{b} \Leftrightarrow \bar{c} = x\bar{a} + y\bar{b}$ for some scalars x,y.

Conversely, suppose that $\bar{c} = x\bar{a} + y\bar{b}$ for some scalars x,y

$\Rightarrow \bar{c}$ lies in the plane generated by $\bar{a}, \bar{b} \Rightarrow \bar{a}, \bar{b}, \bar{c}$ are coplanar.

Hint: **Determination of 3 vectors to be coplanar**: Find a unique pair of reals x,y such that any one of the vectors is a linear combination of the other two vectors (which should be non collinear). Equating the corresponding coefficients of the 3 components in LHS and RHS, we get 3 equations. If the values of x,y obtained by solving any 2 equations, are satisfied by the other equation then the given 3 vectors are coplanar.

Theorem 1.2: Three vectors $\bar{a}, \bar{b}, \bar{c}$ are coplanar $\Leftrightarrow x\bar{a} + y\bar{b} + z\bar{c} = \bar{0}$ for some scalars x,y,z not all zero.

Proof: Suppose $\bar{a}, \bar{b}, \bar{c}$ are coplanar.

Case (i) If $\bar{a} = \bar{0}$ then $x(\bar{a}) + 0(\bar{b}) + 0(\bar{c}) = \bar{0}$, , for some non-zero x

Case (ii) If $\bar{a} \neq \bar{0}$ and $\bar{a}, \bar{b}$ are collinear

Then $\bar{b} = t\bar{a}$ for some $t \in R$, $\Rightarrow t\bar{a} + (-1)\bar{b} + 0(\bar{c}) = \bar{0}$ for $-1 \neq 0$, $t \in R$

Case (iii) if $\bar{a} \neq \bar{0}$ and $\bar{a}, \bar{b}$ are non collinear.

Then $\bar{a}, \bar{b}$ generate a plane and since $\bar{a}, \bar{b}, \bar{c}$ are coplanar $\Rightarrow$ **c** lies in that plane.

$\Rightarrow x\bar{a} + y\bar{b} = \bar{c}$ for some scalars.

$\Rightarrow x\bar{a} + y\bar{b} + (-1)\bar{c} = \bar{0}$ for $-1 \neq 0$, $x, y \in R$

Conversly, suppose that $x\bar{a} + y\bar{b} + z\bar{c} = \bar{0}$ for some scalar x,y,z not all zero.

Let $z \neq 0$, now $x\bar{a} + y\bar{b} + z\bar{c} = \bar{0} \Rightarrow z\bar{c} = -x\bar{a} - y\bar{b}$

$$\Rightarrow \bar{c} = -\left(\frac{x}{z}\right)\bar{a} + \left(\frac{-y}{z}\right)\bar{b} \Rightarrow \bar{a}, \bar{b}, \bar{c} \text{ are coplanar}$$

similarly if $x \neq 0$ or $y \neq 0$ then also we can show that $\bar{a}, \bar{b}, \bar{c}$ are coplanar.

Conclusion 1: Three vectors $\bar{a}, \bar{b}, \bar{c}$ are coplanar $\Leftrightarrow$ $\bar{a}, \bar{b}, \bar{c}$ are linearly dependent.

Conclusion 2: 3 vectors $\bar{a}, \bar{b}, \bar{c}$ are non coplanar $\Leftrightarrow x\bar{a} + y\bar{b} + z\bar{c} = \bar{0} \Rightarrow x=0, y=0, z=0$

Theorem on Condition for coplanarity of 4 points

Theorem 2: Four points A,B,C,D with PVs $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ are coplanar $\Leftrightarrow x\bar{a} + y\bar{b} + z\bar{c} + w\bar{d} = \bar{0}$ for some scalars x,y,z,w not all zero and x+y+z+w=0

Proof: Suppose the 4 points A,B,C,D are coplanar and O be the origin of reference.

$\Rightarrow$ the vectors $\overline{AB}, \overline{AC}, \overline{AD}$ are coplanar.

$\Rightarrow s\overline{AB} + t\overline{AC} = \overline{AD}$ for some scalars s,t

$\Rightarrow s(\overline{OB} - \overline{OA}) + t(\overline{OC} - \overline{OA}) = \overline{OD} - \overline{OA}$

$\Rightarrow s(\bar{b} - \bar{a}) + t(\bar{c} - \bar{a}) = \bar{d} - \bar{a}$

$\Rightarrow s\bar{b} - s\bar{a} + t\bar{c} - \bar{d} + \bar{a} = \bar{0}$

$\Rightarrow (1 - s - t)\bar{a} + s\bar{b} + t\bar{c} + (-1)\bar{d} = \bar{0}$ which is in the form $x\bar{a} + y\bar{b} + z\bar{c} + w\bar{d} = \bar{0}$ for $w = -1 \neq 0$

comparing this with $x\bar{a} + y\bar{b} + z\bar{c} + w\bar{d} = \bar{0}$

we get x=1–s–t, y=s, z=t, w=–1

$\therefore$ x+y+z+w=1–s–t+s+t–1=0

Conversly, suppose that $x\bar{a} + y\bar{b} + z\bar{c} + w\bar{d} = \bar{0}$ for some scalars x,y,z,w not all zero such that x+y+z+w=0.

Now, x+y+z+w=0 $\Rightarrow$ x=–y–z–w for some y,z,w not all zero.

$\therefore$ $x\bar{a} + y\bar{b} + z\bar{c} + w\bar{d} = \bar{0} \Rightarrow (-y - z - w)\bar{a} + y\bar{b} + z\bar{c} + w\bar{d} = \bar{0} \Rightarrow -y\bar{a} - z\bar{a} - w\bar{a} + y\bar{b} + z\bar{c} + w\bar{d} = \bar{0}$

$\Rightarrow y(\bar{b} - \bar{a}) + z(\bar{c} - \bar{a}) + w(\bar{d} - \bar{a}) = \bar{0}$

$\Rightarrow y(\overline{OB} - \overline{OA}) + z(\overline{OC} - \overline{OA}) + w(\overline{OD} - \overline{OA}) = \bar{0}$

$\Rightarrow y(\overline{AB}) + z(\overline{AC}) + w(\overline{AD}) = \bar{0}$ for some y,z,w not all zero.

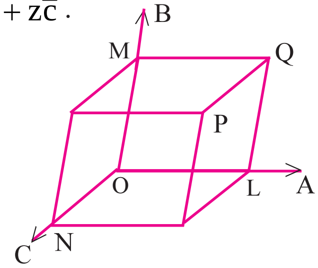
$\Rightarrow$ the three vectors $\overline{AB}, \overline{AC}, \overline{AD}$ are coplanar.

$\Rightarrow$ the 4 points A,B,C,D are coplanar.

A conclusion: Four points $A(\bar{a}), B(\bar{b}), C(\bar{c}), D(\bar{d})$ are coplanar $\Rightarrow$ the vectors $\overline{AB}, \overline{AC}, \overline{AD}$ are coplanar.

Theorem on the resolution of a vector in space

Theorem 3: If $\bar{a}, \bar{b}, \bar{c}$ are 3 non coplanar vectors and $\bar{r}$ is any vector then there exists a unique traid of reals x,y,z such that $\bar{r} = x\bar{a} + y\bar{b} + z\bar{c}$.



Proof: Let P be any point in space and O be the origin of reference then

$\bar{a} = \overline{OA}, \bar{b} = \overline{OB}, \bar{c} = \overline{OC}, r = \overline{OP}$

Let L,M,N be the points of intersection of the planes parallel to COB, AOC, BOA through P with the lines OA, OB, OC respectively.,

Let the line parallel to OC meets the plane BOA at Q.

Also, the pairs of vectors $\overline{OL} \,\&\, \overline{OA}, \overline{OM} \,\&\, \overline{OB}, \overline{ON} \,\&\, \overline{OC}$ are collinear

$\Rightarrow \overline{OL} = x\overline{OA}, \overline{OM} = y\overline{OB}, \overline{ON} = z\overline{OC}$ for some real scalars x,y,z.

Now, $\bar{r} = \overline{OP} = \overline{OL} + \overline{LQ} + \overline{QP} = \overline{OL} + \overline{OM} + \overline{ON} = x\overline{OA} + y\overline{OB} + z\overline{OC} = x\bar{a} + y\bar{b} + z\bar{c}$.

Uniqueness Part:

for if $\exists$ another set of scalar x_1, y_1, z_1 such that $\bar{r} = x_1\bar{a} + y_1\bar{b} + z_1\bar{c}$ then

we have $\bar{r} = x\bar{a} + y\bar{b} + z\bar{c} = x_1\bar{a} + y_1\bar{b} + z_1\bar{c}$.

$\Rightarrow (x - x_1)\bar{a} + (y - y_1)\bar{b} + (z - z_1)\bar{c} = \bar{0}$

$\Rightarrow x-x_1=0, y-y_1=0, z-z_1=0$ ($\because \bar{a}, \bar{b}, \bar{c}$ are non coplanar, hence linearly independent)

$\Rightarrow x=x_1, y=y_1, z=z_1$

$\therefore \bar{r} = x\bar{a} + y\bar{b} + z\bar{c}$ for some unique traids of reals x,y,z.

Components of a vector

Components: If $\bar{r} = x\bar{a} + y\bar{b} + z\bar{c}$ then the unique triad of reals x,y,z are called **scalar components** of $\bar{r}$ and $x\bar{a}, y\bar{b}, z\bar{c}$ are called **vector components** w.r.t the **base vectors** $\bar{a}, \bar{b}, \bar{c}$.

Note: In place of $\bar{a}, \bar{b}, \bar{c}$ if we consider the right handed system of orthonormal vector triad $\bar{i}, \bar{j}, \bar{k}$ (i.e., $\bar{i}, \bar{j}, \bar{k}$ are (i)unit vectors (ii)mutually perpendicular (iii)in right handed system) as base vectors then $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$. Here, the components (x,y,z) determine the coordinates of the point P(x,y,z) in space.

In this regard the vector $\overline{OP} = x\bar{i} + y\bar{j} + z\bar{k}$ is denoted by P(x,y,z) and vice versa.

Linearly Dependent and Linearly Independent Vectors

Note: The concepts of Linearly dependent vectors and linearly independent vectors are not included in the prescribed syllabus.

1.1. **Linearly dependent vectors (L.D vectors)**:

The vectors $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n$ are said to be linearly dependent vectors if a linear combination $x_1\bar{a}_1 + x_2\bar{a}_2 + \ldots + x_n\bar{a}_n = \bar{0}$ is possible for some scalars $x_1, x_2, \ldots, x_n$ **not all zero**.

Rem 1: Null vector $\{\bar{0}\}$ is a linearly dependent vector.

Rem 2: Any two collinear vectors are linearly dependent vectors.

Rem 3: Any three coplanar vectors are linearly dependent vectors.

Rem 4: Any four vectors are L.D.

Rem 5: The vectors represented by the sides of a triangle are linearly dependent vectors.

4. ADDITIONAL QS ON VECTOR EQUATIONS

1. Find the point of intersection of the line $\bar{r}=2\bar{a}+\bar{b}+t(\bar{b}-\bar{c})$ and the plane $\bar{r}=\bar{a}+x(\bar{b}+\bar{c})+y(\bar{a}+2\bar{b}-\bar{c})$ where $\bar{a},\bar{b},\bar{c}$ are non-coplanar vectors.

Sol: The equation of the given lines is $\bar{r}=2\bar{a}+\bar{b}+t(\bar{b}-\bar{c})$, $t\in R$ .....(1)

The equation of the given plane is $\bar{r}=\bar{a}+x(\bar{b}+\bar{c})+y(\bar{a}+2\bar{b}-\bar{c})$, $x,y\in R$....(2)

If the given line and the plane intersect each other, then from (1) & (2), we have

$2\bar{a}+\bar{b}+t(\bar{b}-\bar{c})=\bar{a}+x(\bar{b}+\bar{c})+y(\bar{a}+2\bar{b}-\bar{c})$

$\Rightarrow 2\bar{a}+\bar{b}(1+t)-t\bar{c}=\bar{a}(1+y)+\bar{b}(x+2y)+\bar{c}(x-y)$

Equating the corresponding coefficients, we get $1+y=2\Rightarrow y=1$

$x+2y=1+t \Rightarrow x+2(1)=1+t \Rightarrow x+2=1+t \Rightarrow x=t-1$ ...(3)

Also, $x-y=-t \Rightarrow x-1=-t \Rightarrow x=-t+1$ .....(4)

From (3) & (4), we have $t-1=-t+1 \Rightarrow 2t=2 \Rightarrow t=1$

Substituting this value of t=1 in (1), we get

the point of intersection as $2\bar{a}+\bar{b}+1(\bar{b}-\bar{c})=2\bar{a}+2\bar{b}-\bar{c}$

---

2. Find the vector equation of the plane passing through the points $4\bar{i}-3\bar{j}-\bar{k}$, $3\bar{i}+7\bar{j}-10\bar{k}$ and $2\bar{i}+5\bar{j}-7\bar{k}$ and show that the point $\bar{i}+2\bar{j}-3\bar{k}$ lies in the plane.

Sol: Let $A(\bar{a})=4\bar{i}-3\bar{j}-\bar{k}$, $B(\bar{b})=3\bar{i}+7\bar{j}-10\bar{k}$, $C(\bar{c})=2\bar{i}+5\bar{j}-7\bar{k}$, $D(\bar{d})=(\bar{i}+2\bar{j}-3\bar{k})$

Equation of the plane passing through the points $A(\bar{a}),B(\bar{b}),C(\bar{c})$ is

$\bar{r}=(1-s-t)\bar{a}+s\bar{b}+t\bar{c}$ $s,t\in R$

$\Rightarrow \bar{r}=(1-s-t)(4\bar{i}-3\bar{j}-\bar{k})+s(3\bar{i}+7\bar{j}-10\bar{k})+t(2\bar{i}+5\bar{j}-7\bar{k})$

Substituting $D(\bar{i}+2\bar{j}-3\bar{k})$ in the above equation in place of $P(\bar{r})$, we have

$\bar{i}+2\bar{j}-3\bar{k}=(1-s-t)(4\bar{i}-3\bar{j}-\bar{k})+s(3\bar{i}+7\bar{j}-10\bar{k})+t(2\bar{i}+5\bar{j}-7\bar{k})$

Equating the corresponding coefficients of $\bar{i},\bar{j},\bar{k}$ we get

$(4-4s-4t)+3s+2t=1\Rightarrow s+2t=3$.....(1)

$(-3+3s+3t)+7s+5t=2\Rightarrow 10s+8t=5$....(2)

$(-1+s+t)-10s-7t=-3\Rightarrow 9s+6t=2$.....(3)

Solving (1) & (2) we get $s=-7/6$, $t=25/12$

Substituting these values of s,t in (3), we have $9\left(-\frac{7}{6}\right)+6\left(\frac{25}{12}\right)=-\frac{21}{2}+\frac{25}{2}=\frac{4}{2}=2$

$\therefore$ The point D lies on the plane.