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DIGITAL CONTENT MATERIAL

PARABOLA -INDEX

CONIC SECTIONS

PARABOLA, ELLIPSE, HYPERBOLA

SECTIONS	*No. of periods* (24)	*Weightage in IPE* $2 \times 2 + 3 \times 4 + 1 \times 7 = 23$
Introduction-Conic sections	1	
3. The Parabola	10	2 and 7
4. The Ellipse	8	4 and 4
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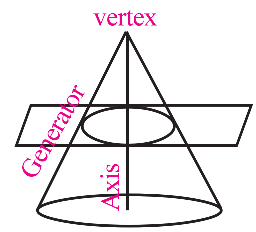
- *The path of a projectile is a parabola.*
- *The 'sixer' hit by a batsman in the Cricket game traces a parabola.*
- *The quadratic equation* $y = ax^2 + bx + c, (\Delta \neq 0)$ *traces the graph of a parabola.*
- *According to Sommerfeld's atomic theory, electrons revolve around the nucleus in elliptical orbits.*
- *All the planets are revolving round the sun in different elliptical orbits. The shapes of the elliptical orbits with a measure of departure from circularity is determined by the eccentricities of the elliptical orbits e.g., the eccentricity of the earth's orbit is 0.02.*
- *The shadow of a lamp casts hyperbola. Open orbits of comets about the sun follow hyperbolas. Cooling towers of power plants are designed in hyperboloid shapes in order to withstand high winds & use minimum material.*
- *The curves Parabola, Ellipse, Hyperbola have certain similar features as well as independent characteristic properties.*
- *In this chapter conic sections, analytic treatment is given to the conics viz., Parabola, Ellipse and Hyperbola.*
- *These conic sections are defined analytically and hence equations in the standard forms are derived.*
- *Various terms such as eccentricity, directrix, focus, vertex, focal chord, latus rectum etc., are explained in the introductory section.*
- *The problem of determining focus, vertex, equation of Latus rectum, directrix, etc., and finding the equations of conics satisfying the given conditions are discussed.*
- *The concepts of tangents, normals and tangential conditions(pole, polar, conjugate points, conjugate lines are not prescribed for IPE but are useful for IIT-JEE)are discussed in connection with the conics.*
- *Parametric treatment is given whereever required.*

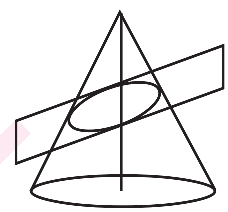
3.0.INTRODUCTION

I. Geometrical treatment to conic sections:

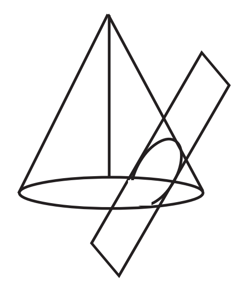
The conic sections are the plane curves obtained from the plane sections, when a right circular cone is cut by a plane section, in different angles and positions. Hence, accordingly we get the shapes of a circle, ellipse, parabola, hyperbola and pair of lines.



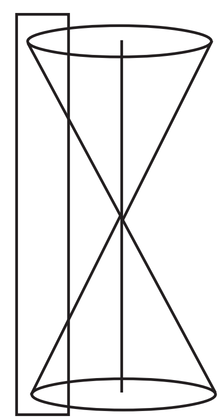
1. If the cutting plane is perpendicular to the axis of the cone then we get the shape of a circle.



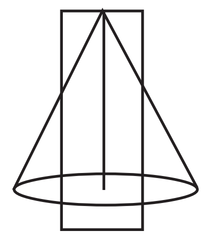
2. If the cutting plane makes an acute angle (should not be parallel generator of the cone) with the axis of the cone then we get the shape of an ellipse



3. If the cutting plane is parallel to a generator of the cone then we get the shape of a parabola.



4. If the cutting plane is parallel to the axis of a full cone (double cone) then we get the shape of a hyperbola.

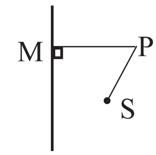


5. If the cutting plane passes through the vertex of the cone then we get a pair of lines intersecting at the vertex.

II. Analytical treatment to conic sections:

1. ***Analytical definition of conic section***:

A conic section is the locus of a point in a plane such that its distance from a fixed point bears a constant ratio to its distance from a fixed line.



The **fixed point** is called the **focus** and it is denoted by **S**
The **fixed line** is called the **directrix** of the conic
The **constant ratio** is called the **eccentricity** and it is denoted by **e**
PM denotes the **perpendicular distance** from point P on the conic to the directrix.

Thus, the equation of the conic is the locus of P(x,y) such that $\frac{SP}{PM} = e \Rightarrow SP^2 = e^2 PM^2$

The above property is known as **focus directrix property** of the conic.

2. If **e=1** then the conic is called a **parabola**
If **0<e<1** then the conic is called an **ellipse**
If **e>1** then the conic is called a **hyperbola**.

3. Terminology of conic sections:

1. ***Principle axis***: The line perpendicular to the directrix and passing through the focus is called the principle axis (or axis) of the conic. Principle axis is a line of symmetry to the conic.
2. ***Vertices***: The points of intersection of the conic with its axis are called the vertices of the conic.Parabola has only one vertex where as ellipse and hyperbola have more than one vertex
3. ***Centre***: The mid point of the line segment joining the vertices of the conic is called the centre

   Conics having centres are called central conics. Thus ellipse and hyperbola are central conics. In the parabola, the vertex itself is treated as its centre.
4. ***Chord***: The line segment joining any 2 points on the conic is called the chord of the conic.
5. ***Focal chord***: Any chord passing through the focus of the conic is called the focal chord of the conic.
6. ***Latus rectum***: The focal chord which is perpendicular to the principle axis is called the latus rectum of the conic.
7. ***Double ordinate***: Any chord perpendicular to the axis and intersecting the conic is called double ordinate of the conic.
8. ***Diameter***: The locus of the mid points of a family of parallel chords, is called a diameter of the conic.

   In the central conics, the diameter passes through the centre of the conic, where as a diameter of a parabola is a line parallel to the axis of the parabola. Two diameters are said to be **conjugate diameters** when each bisects all chords parallel to the other.
9. ***Pole and polar***: The polar of a point w.r.to a conic is the straight line containing the points of intersection of the tangents drawn at the ends of the chords passing through that point. The point is called the pole of the conic.
10. ***Standard form of a conic***: A conic is said to be in standard form if its principle axis coincides with the x-axis and centre coincides with the origin.

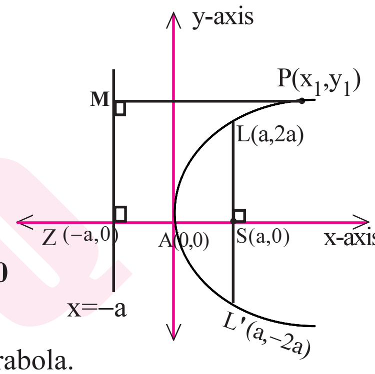
4. **Equation of the conic**: The general equation of second degree in x and y i.e.,
$ax^2+2hxy+by^2+2gx+2fy+c=0$ represents the equation of a conic.

3. PARABOLA

SYNOPSIS POINTS

Parabola: The locus of a point in a plane, which moves such that its distance from a fixed point is equal to its distance from a fixed line, is called a parabola.

The following terminology holds true w.r.to the parabola $y^2 = 4ax$ in the standard form:



1. The **focus** of the parabola $y^2 = 4ax$ is **S(a,0)**

2.1. The equation of the **directrix** of the parabola is **x= –a**

2.2. The **foot of the directrix** is **Z(–a,0)**

3. The **vertex** of the parabola is **A(0,0)**

4. The **axis** of the parabola is the **x-axis** and its equation is **y=0**

5. The **tangent** at the vertex is the **y-axis** and its equation is **x=0**

6.1. The line segment passing through the focus S(a,0) and perpendicular to the axis is called the **latusrectum** of the parabola.

6.2. The **equation of the latusrectum** is **x=a**

6.3. The **ends of the latusrectum** of the parabola are **L=(a,2a) and L' =(a,–2a)**

6.4. The **length of the latusrectum LL'** of the parabola $y^2=4ax$ is **4a**

6.5. LS=*l* is known as a **semi latusrectum** of the parabola and its length is **2a**

7.1. Any chord passing through the focus S(a,0) is known as the **focal chord** of the parabola

7.2. The **focal distance** of the point $P(x_1,y_1)$ on the parabola $y^2=4ax$ is $\mathbf{SP=x_1+a}$

8. Notation: $S=y^2-4ax$; $S_1=y_1y-2a(x_1+x)$; $S_{11}=y_1^2-4ax_1$; $S_{12}=y_1y_2-2a(x_1+x_2)$

9. Relative positions of a point and parabola: The point $P(x_1,y_1)$ w.r.to the parabola S=0 lies

(i) on the parabola $\Leftrightarrow S_{11}=0$; (ii) inside the parabola $\Leftrightarrow S_{11}<0$

(iii) outside the parabola $S=0 \Leftrightarrow S_{11}>0$

10. The parametric equations of the parabola $y^2=4ax$ with parameter t are $x=at^2$, $y=2at$

The point $(at^2, 2at)$ is simply denoted by the parameter t

11.1. The equation of the tangent drawn at a point $P(x_1,y_1)$ on the parabola S=0 is $S_1=0$

11.2. The tangential condition for the line y=mx+c and the parabola $y^2=4ax$ is $c = \frac{a}{m}$

11.3. The equation of the tangent to the parabola $y^2=4ax$ having slope m is $y = mx + \frac{a}{m}$

11.4. The equation of the tangent to the parabola $y^2=4ax$ at the point ‘t’ is $yt=x+at^2$.

12.1. The equation of the normal at $P(x_1,y_1)$ on the parabola $y^2=4ax$ is $y - y_1 = -\frac{y_1}{2a}(x - x_1)$

12.2. The equation of the normal to the parabola $y^2=4ax$ at the point t is $y+xt=2at+at^3$

ILLUSTRATIVE RESULTS WITH CERTAIN PROOFS

1. **Notation:** The following notation is adopted throughout this chapter.

$S = y^2 - 4ax$	$S = x^2 - 4ay$
$S_1 = y_1y - 2a(x_1 + x)$	$S_1 = x_1x - 2a(y_1 + y)$
$S_{11} = y_1^2 - 4ax_1$	$S_{11} = x_1^2 - 4ay_1$
$S_{12} = y_1y_2 - 2a(x_1 + x_2)$	$S_{12} = x_1x_2 - 2a(y_1 + y_2)$

2. **Relative positions of a point $P(x_1,y_1)$ and the parabola $S=y^2-4ax=0$**

   (i) The point $P(x_1,y_1)$ lies on the parabola $S=0 \Leftrightarrow S_{11}=0$

   (ii) The point $P(x_1,y_1)$ lies inside the parabola $S=0 \Leftrightarrow S_{11}<0$

   (iii) The point $P(x_1,y_1)$ lies outside the parabola $S=0 \Leftrightarrow S_{11}>0$

3. **The equation of the chord joining the points (x_1,y_1) and (x_2,y_2) on the parabola $S=0$ is $S_1+S_2=S_{12}$**

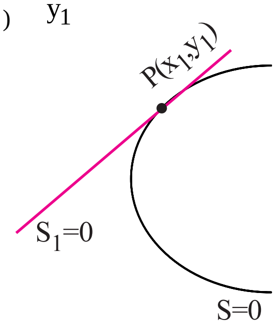
4. ***Theorem*: The equation of the tangent drawn at a point $P(x_1,y_1)$ on the parabola $S=0$ is $S_1=0$**

Proof: Consider the standard equation of the parabola $S=y^2-4ax=0$

differentiating the above equation w.r.t x, we get $2y\frac{dy}{dx}-4a=0 \Rightarrow \frac{dy}{dx}=\frac{2a}{y}$

$\therefore$ The slope of the tangent at $P(x_1,y_1)$ on $S=0$ is $m=\left(\frac{dy}{dx}\right)_{(x_1,y_1)}=\frac{2a}{y_1}$

$\therefore$ The equation of the tangent at $P(x_1,y_1)$ with slope $\frac{2a}{y_1}$ is



$y-y_1=\frac{2a}{y_1}(x-x_1) \Rightarrow y_1(y-y_1)=2a(x-x_1)$

$\Rightarrow yy_1-y_1^2=2ax-2ax_1 \Rightarrow yy_1-2ax=y_1^2-2ax_1$

adding $-2ax_1$ both sides $yy_1-2ax-2ax_1=y_1^2-2ax_1-2ax_1$

$\Rightarrow yy_1-2a(x+x_1)=y_1^2-4ax_1$

$\Rightarrow yy_1-2a(x_1+x)=0$ $[\because (x_1,y_1)$ lies on $y^2-4ax=0 \Rightarrow y_1^2-4ax_1=0]$

$\Rightarrow S_1=0$

$\therefore$ The equation of the tangent at (x_1,y_1) on the parabola $S=0$ is $S_1=0$

Ex: Find the equation of the tangent at (3,6) on the parabola $y^2=12x$

Sol: The equation of the tangent at (x_1,y_1) on the parabola $y^2=4ax$ is $y_1y-2a(x_1+x)=0$

The equation of the tangent at (3,6) on the parabola $y^2=12x$ is $6y-6(3+x)=0$

$\Rightarrow 6y-18-6x=0$

$\Rightarrow y-3-x=0 \Rightarrow x-y+3=0$

5. *Theorem*: The equation of the normal at $P(x_1,y_1)$ on the parabola $y^2=4ax$ is

$$y-y_1=-\frac{y_1}{2a}(x-x_1)$$

Proof : The slope of the tangent $yy_1-2a(x_1+x)=0$ at $P(x_1,y_1)$ on the parabola is $\frac{2a}{y_1}$

$\Rightarrow$ The slope of the normal at $P(x_1,y_1)$ on the parabola is $-\frac{y_1}{2a}$

$\therefore$ The equation of the normal at $P(x_1,y_1)$ with slope $-\frac{y_1}{2a}$ is $y-y_1=-\frac{y_1}{2a}(x-x_1)$

Ex: Find the equation of the normal at (1, 2) on the parabola $y^2 = 4x$

Sol: The equation of the normal at $P(x_1,y_1)$ on $y^2=4ax$ is $y-y_1=-\frac{y_1}{2a}(x-x_1)$

$\therefore$ The equation of the normal at (1,2) on $y^2=4x$ is $y-2=\frac{-2}{2}(x-1)$

$\Rightarrow y-2=-1(x-1) \Rightarrow x-1+y-2=0 \Rightarrow x+y-3=0$

6. Relative positions of the parabola $y^2 = 4ax$, $a > 0$ and the line $y = mx + c$, $m \neq 0$:

Solving $y^2=4ax$ and $y=mx+c$, we have $(mx+c)^2=4ax$.

$\Rightarrow m^2x^2+2x(mc-2a)+c^2=0$

The above equation is a quadratic equation in m, hence it has 2 roots which are distinct real, equal or imaginary according as the sign of its discriminant.

Now, $\Delta=4(mc-2a)^2-4m^2c^2=16a(a-mc)$

Case (i): If $\Delta>0$ then $16a(a-mc)>0 \Rightarrow a-mc>0$ ($\because$ $a>0$) $\Rightarrow a>mc \Rightarrow c<\frac{a}{m}$

Hence if $c<\frac{a}{m}$ then the line $y=mx+c$ intersects the parabola $y^2=4ax$.

Case (ii): If $\Delta<0$ then $c>\frac{a}{m}$ and here the line $y=mx+c$ does not meet the parabola.

Case (iii): If $\Delta=0$ then $c=\frac{a}{m}$ and here the line $y=mx+c$ touches the parabola $y^2=4ax$

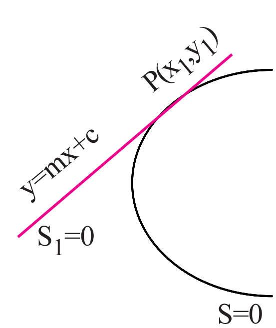
7. *Theorem*: The condition for the line $y = mx + c$ to be a tangent to the parabola $y^2 = 4ax$ is $c=\frac{a}{m}$

Proof: Let $P(x_1,y_1)$ be the point of contact of the parabola $S = y^2-4ax = 0$ and the line $y=mx+c$

Now, the equation of the tangent at $P(x_1,y_1)$ on $S = 0$ is $S_1 = 0$

$\Rightarrow y_1y-2a(x_1+x)=0 \Rightarrow yy_1=2ax+2ax_1$

Now, comparing the above equation with $y=mx+c$, we have



$$\frac{y_1}{1}=\frac{2a}{m}=\frac{2ax_1}{c} \Rightarrow y_1=\frac{2a}{m} \text{ and } x_1=\frac{c}{m} \Rightarrow P(x_1,y_1)=\left(\frac{c}{m},\frac{2a}{m}\right)$$

But the point $P(x_1,y_1)$ lies on the line $y=mx+c$

$$\Rightarrow \frac{2a}{m}=m\left(\frac{c}{m}\right)+c \Rightarrow \frac{2a}{m}=2c \Rightarrow c=\frac{a}{m}$$

Cor 1: The condition for the line lx+my+n=0 to touch the parabola $y^2=4ax$ is $ln=am^2$

Cor 2: The condition for the line y=mx+c to be a tangent to the parabola $x^2=4ay$ is $c=-am^2$

8. Theorem : The equation of the tangent to the parabola $y^2 = 4ax$ having slope m is

$$y = mx + \frac{a}{m}$$

Proof: Consider the line y = mx + c and the parabola $y^2 = 4ax$

The equation y=mx+c represents a family of parallel lines for the arbitrary constant c and for fixed m. Among these only one line touch the ellipse satisfying the tangential condition c=a/m

$\therefore$ the equation of the tangent with slope m to the parabola is $y = mx + \frac{a}{m}$

Ex: Show that the line x–3y+9=0 touches the parabola $y^2=4x$

Sol: The given line is $x-3y+9=0 \Rightarrow 3y = x + 9 \Rightarrow y = \frac{1}{3}x + 3 \Rightarrow m = \frac{1}{3}$ and c=3

Also $y^2=4x \Rightarrow 4a=4 \Rightarrow a=1$

Now, applying the tangential condition $c = \frac{a}{m}$, we have $3 = \frac{1}{1/3} \Rightarrow 3 = 3$

$\therefore$ the tangential condition is satisfied

$\therefore$ the given line x–3y+9=0 touches the parabola $y^2=4x$

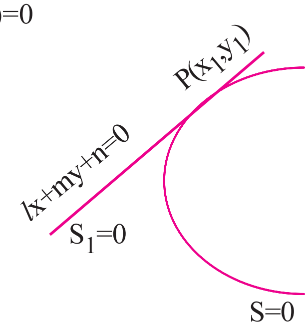
9. Theorem: The point of contact of the tangent line lx+my+n=0 and the parabola $y^2=4ax$ is $\left(\frac{n}{l}, \frac{-2am}{l}\right)$

Proof: Let $P(x_1,y_1)$ be the point of contact of $S=y^2-4ax=0$ and the tangent line $lx+my+n=0$......(1)

The equation of the tangent at $P(x_1,y_1)$ on S=0 is $S_1=yy_1-2a(x+x_1)=0$

$\Rightarrow 2ax-y_1y+2ax_1=0$ .... (2)

Now (1) and (2) represent the equation of the same line



$$\therefore \frac{2a}{l} = \frac{-y_1}{m} = \frac{2ax_1}{n}$$

Now, $\frac{2a}{l} = \frac{2ax_1}{n} \Rightarrow x_1 = \frac{n}{l}$ and $\frac{-y_1}{m} = \frac{2a}{l} \Rightarrow y_1 = -\frac{2am}{l}$

$$\therefore P(x_1,y_1) = \left(\frac{n}{l}, \frac{-2am}{l}\right)$$

Cor 1: The point of contact of the tangent line y=mx+c, m ≠ 0 and the parabola $y^2=4ax$ is

$$\left(\frac{a}{m^2}, \frac{2a}{m}\right) \text{ or } \left(\frac{c}{m}, 2c\right)$$

Cor 2: The point of contact of the tangent line lx + my + n = 0 and the parabola $x^2 = 4ay$ is

$$\left(\frac{-2al}{m}, \frac{n}{m}\right)$$

Ex: Find the point of contact of the tangent x+y+3=0 and the parabola $y^2=12x$

Sol: The given tangent is x+y+3=0 $\Rightarrow$ y= -x-3 $\Rightarrow$ m=-1, c=-3

$\therefore$ the point of contact is $\left(\frac{c}{m}, 2c\right) \Rightarrow \left(\frac{-3}{-1}, 2(-3)\right) = (3,-6)$

(or)

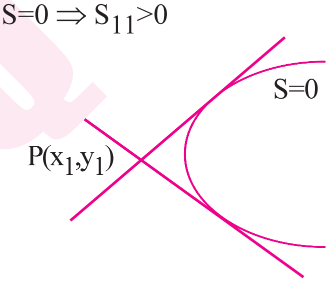
comparing x+y+3=0 with $lx+my+n=0$ we get $l=1$, m=1, n=3

and comparing $y^2=12x$ with $y^2=4ax$ we get 4a=12 $\Rightarrow$ a=3

$\therefore$ The point of contact is $\left(\frac{n}{l}, -\frac{2am}{l}\right) \Rightarrow \left(\frac{3}{1}, \frac{-2(3)(1)}{1}\right) = (3,-6)$

10. Theorem: Two tangents can be drawn from an external point $P(x_1,y_1)$ to the parabola $y^2=4ax$ and hence if m_1,m_2 are the slopes of the two tangents then $m_1+m_2=\frac{y_1}{x_1}$ & $m_1m_2=\frac{a}{x_1}$

Proof : Given that $P(x_1, y_1)$ is an external point to the parabola S=0 $\Rightarrow S_{11}>0$



Let $y = mx + \frac{a}{m}$ be a tangent to the parabola with slope m.

If this tangent pass through $P(x_1,y_1)$ then

$y_1 = mx_1 + \frac{a}{m} \Rightarrow y_1m = m^2x_1 + a$

$\Rightarrow m^2x_1 - y_1m + a = 0$.......(1)

(1) is a quadratic equation in m and the discriminent of (1) is $\Delta = y_1^2 - 4ax_1 > 0 \because S_{11}>0$

$\Rightarrow$ there exist two different real values for m and hence two tangents can be drawn from an external point to the parabola.

We take m_1,m_2 as the roots of (1), which are nothing but the slopes of the two tangents then the sum of the roots $m_1 + m_2 = \frac{y_1}{x_1}$ [from (1)] and the product of the roots $m_1m_2 = \frac{a}{x_1}$

Rem: If θ is the angle between pair of tangents drawn from $P(x_1,y_1)$ to $S=y^2-4ax=0$ then $\tan\theta=\frac{\sqrt{S_{11}}}{x_1+a}$

Ex: Show that two tangents can be drawn from (2, 3) to the parabola $y^2 = 4x$ and also find the sum and product of the slopes of the tangents.

Sol: Let the given point be $(2, 3) = (x_1, y_1)$

Now, w.r.t the parabola $S=y^2-4x=0$, we have $S_{11}=(3)^2-4(2) = 9 - 8 = 1>0$

$\Rightarrow$ the given point (2,3) is an external point to the parabola $y^2=4x$

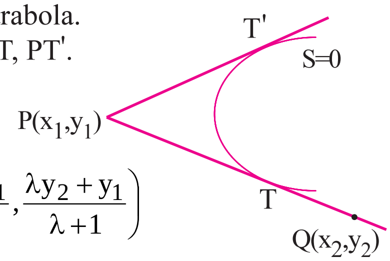
$\therefore$ two tangents can be drawn from (2,3) to $y^2=4x$

Also, the sum of the slopes of the tangents $m_1 + m_2 = \frac{y_1}{x_1} = \frac{3}{2}$

the product of the slopes of the tangents $m_1m_2 = \frac{a}{x_1} = \frac{1}{2}$

11. Theorem: The combined equation of the pair of tangents drawn from an external point $P(x_1,y_1)$ to the parabola $S=y^2-4ax=0$ is $S_1^2=S_{11}S$

Proof: The equation of the given parabola is $S = y^2-4ax = 0$ & $P(x_1, y_1)$ is the given external point from which a pair of tangents PT, PT' are drawn to the parabola.
Let $Q(x_2, y_2)$ be a variable point on the pair of tangents PT, PT'.
Then PQ will touch the parabola at T or T'



Let that point T divide PQ in the ratio $\lambda:1$.

Now, by the ratio formula the coordinates of $T=\left(\frac{\lambda x_2+x_1}{\lambda+1},\frac{\lambda y_2+y_1}{\lambda+1}\right)$

But the above point lies on the parabola $y^2=4ax \Rightarrow \left(\frac{\lambda y_2+y_1}{\lambda+1}\right)^2=4a\left(\frac{\lambda x_2+x_1}{\lambda+1}\right)$

$(\lambda y_2+y_1)^2=4a(\lambda+1)(\lambda x_2+x_1)$, which is a quadratic in λ and it can be written in the form

$\lambda^2(y_2^2-4ax_2)+2\lambda(y_1y_2-2a(x_1+x_2))+y_1^2-4ax_1=0$

This can be written in the notational form $\lambda^2S_{22}+2\lambda S_{12}+S_{11}=0$ .....(1)
PQ touches the parabola at only one point $\Rightarrow$ The roots of λ must be equal
$\Rightarrow 4(S_{12})^2=4S_{22}S_{11}$ (from (1) $\Delta=b^2-4ac=0 \Rightarrow b^2=4ac$)
Hence, the locus of $Q(x_2,y_2)$ is $(S_1)^2=S.S_{11}$

$\Rightarrow [yy_1-2a(x+x_1)]^2=(y_1^2-4ax_1)(y^2-4ax)$
Thus, the equation of the pair of tangents, from (x_1, y_1) to the parabola S=0 is $(S_1)^2=S.S_{11}$

Ex: Find the combined equation of the pair of tangents drawn from P(−3,2) to the parabola $y^2=4x$. Also find the separate equations of the pair of tangents.

Sol: The combined equation of the pair of tangents drawn from an external point to the parabola S=0 is $S_1^2=S_{11}S$

$\therefore$ The combined equation of the pair of tangents drawn from the external point (-3,2) to the parabola $S=y^2-4x=0$ is $S_1^2=S_{11}S$

$\Rightarrow [2y-2((-3)+x)]^2=(2^2-4(-3))(y^2-4x)$

$\Rightarrow (2(y-(-3+x)))^2=16(y^2-4x)$

$\Rightarrow 4(y+3-x)^2=16(y^2-4x) \Rightarrow (y+3-x)^2=4(y^2-4x)$

$\Rightarrow x^2+y^2+9-2xy+6y-6x=4y^2-16x$

$\Rightarrow x^2-3y^2-2xy+10x+6y+9=0$

Let the line passing through (−3,2) has slope m. Then the equation of the line is $y-2=m(x+3)$ ..(1)

$\Rightarrow y=mx+3m+2$

For this line to be a tangent to the parabola $y^2=4x$, the tangential condition c=a/m must be satisfied

$\Rightarrow 3m+2=1/m \Rightarrow 3m^2+2m-1=0 \Rightarrow m=1/3, -1$

Substituting these values in (1) we get, the equations of the tangents as $x+y+1=0$, $x-3y+9=0$

12. Theorem: The equation of the chord of contact of $P(x_1,y_1)$ w.r.t the parabola S=0 is $S_1=0$

Ex: Find the equation of the chord of contact of the point (2,5) w.r.t the parabola $y^2=8x$

Sol: The equation of the chord of contact of (2,5) w.r.t S=0 is $S_1=0$

$\Rightarrow 5y=4(x+2) \Rightarrow 4x-5y+8=0$

13. Theorem: The equation of the chord of the parabola S=0 having $P(x_1,y_1)$ as its midpoint is $S_1=S_{11}$.

Eg: Find the equation of the chord of the parabola $y^2=4x$ having (3,–2) as its midpoint.

Sol: The equation of the chord with (x_1,y_1) as its midpoint to the parabola S=0 is $S_1=S_{11}$

$\Rightarrow -2y-2(x+3)=(-2)^2-4(3)$

$\Rightarrow -2y-2x-6=-8 \Rightarrow 2x+2y-2=0 \Rightarrow x+y-1=0$

14. Theorem: The equation of the polar of the point $P(x_1,y_1)$ w.r.t the parabola S=0 is $S_1=0$

Eg: Find the polar of (2,3) w.r.t the parabola $y^2=4x$

Sol: The equation of the polar of (2,3) w.r.t $y^2=4x$ is $S_1=0 \Rightarrow 3y-2(x+2)=0 \Rightarrow 2x-3y+4=0$

15. Theorem: The pole of the line lx+my+n=0, (l≠0) w.r.t the parabola $y^2=4ax$ is $\left(\frac{n}{l}, \frac{-2am}{l}\right)$

Eg: Find the pole of x–2y+3=0 w.r.t $y^2=4x$.

Sol: Comparing x–2y+3=0 with the equation $lx+my+n=0$, we get $l=1$, m= –2, n=3

comparing $y^2=4x$ with $y^2=4ax$, we get a=1

$$\therefore \text{Pole} = \left(\frac{n}{l}, \frac{-2am}{l}\right) = \left(\frac{3}{1}, \frac{-2(1)(-2)}{1}\right) = (3,4)$$

16. Theorem: The condition for the points $P(x_1,y_1)$ and $Q(x_2,y_2)$ to be conjugate w.r.t the parabola S=0 is $S_{12}=0$

Ex: Show that (2, 1), (1, 6) are conjugate points w.r.t $y^2 = 4x$

Sol: The polar of (2,1) w.r.t $y^2 = 4x$ is $(1)y = 2(x + 2) \Rightarrow 2x - y + 4 = 0$

Now, substituting (6,1) in the above expression, 2(1)–6+4=0

$\therefore$ (2, 1), (1, 6) are conjugate points w.r.t $y^2 = 4x$

17. Theorem: The condition for the lines $l_1x+m_1y+n_1=0$ and $l_2x+m_2y+n_2=0$ to be conjugate w.r.t the parabola $y^2=4ax$ is $l_1n_2+l_2n_1=2am_1m_2$.

Ex: Show that the lines x–y+5=0, 4x–3y–8=0 are conjugate w.r.to $y^2=8x$

Sol: Comparing the given lines with $l_1x+m_1y+n_1=0$ and $l_2x+m_2y+n_2=0$ we get,

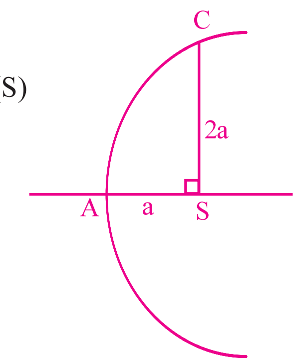
$l_1=1$, $m_1=-1$, $n_1=5$, $l_2=4$, $m_2=-3$, $n_2=-8$ also comparing $y^2=8x$ and $y^2=4ax$ we get a=2

Now, applying the condition $l_1n_2+l_2n_1=2am_1m_2 \Rightarrow 1(-8)+4(5)=2(2)(-1)(-3) \Rightarrow 12=12$

$\therefore$ The given lines are conjugate

ADDITIONAL QUESTIONS WITH SOLUTIONS

1 **A comet moves in a parabolic orbit with the sun as focus. When the comet is $2x10^7$ K.M from the sun, the line from the sun to it makes an angle $\pi/2$ with the axis of the orbit. Find how near the comet comes to the sun.**



Sol: Let the comet (C) traces the parabola $y^2 = 4ax$ and its focus is sun(S)

Now SC = Length of the semi latus rectum = 2a

From the given data, it is clear that $SC = 2 \times 10^7$

$\therefore 2a = 2\times10^7 \Rightarrow a = 10^7$ K.M

The nearest distance between S and C is the focal radius SA = a $=10^7$ K.M

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2 **Find the equation of the parabola whose vertex and focus are on the positive x-axis at a distance 'a' and 'a''from the origin respectively.**

Sol: From the given data, vertex A(h,k) = (a,0),

Focus S = (a',0) $\Rightarrow AS = |a' - a|$

$\therefore$ Equation of the parabola is $(y-k)^2 = 4(AS)(x-h)$

$\Rightarrow (y-0)^2 = 4|a'-a|(x-a)$

$\Rightarrow y^2 = 4|a'-a|(x-a)$

---

3 **Find the locus of the points of trisection of double ordinate of a parabola $y^2 = 4ax(a>0)$.**

Sol: Equation of the parabola is $y^2 = 4ax$

P(x, y) and Q(x, –y) are the ends of the double ordinate

Let T and T' are the points of trisection of double ordinate $\overline{PQ}$

If T divides $\overline{PQ}$ in the ratio 1:2 internally then $T = \left(\frac{1(x)+2(x)}{1+2}, \frac{1(-y)+2(y)}{1+2}\right) = \left(x, \frac{y}{3}\right)$

If T' divides $\overline{PQ}$ in the ratio 2:1 internally then $T' = \left(\frac{2(x)+1(x)}{2+1}, \frac{2(-y)+1(y)}{2+1}\right) = \left(x, \frac{-y}{3}\right)$

Suppose Co-ordinates of the required points T and T' be (x_1, y_1)

$$\Rightarrow y_1 = \pm\frac{y}{3} \Rightarrow y_1^2 = \frac{y^2}{9} \Rightarrow y^2 = 9y_1^2 \Rightarrow 4ax_1 = 9y_1^2$$

Locus of (x_1, y_1) is $9y^2 = 4ax$.

4 **If L and L' are the ends of the latus rectum of the parabola $x^2 = 6y$, find the equations of OL and OL' where 'O' is the origin. Also find the angle between them.**

Sol: Equation of the parabola is $x^2 = 6y$

Comparing $x^2 = 6y$ with $x^2 = 4ay$ we get $4a = 6 \Rightarrow 2a = 3 \Rightarrow a = \frac{3}{2}$

The ends of the latus rectum are L=(2a, a) and L'=(–2a, a) $\Rightarrow L=\left(3,\frac{3}{2}\right), L'=\left(-3,\frac{3}{2}\right)$

We know the equation of the line passing through O(0,0) and (x_1,y_1)

$$\frac{x}{x_1} = \frac{y}{y_1} \Rightarrow x_1 y = y_1 x$$

The equation of line OL passing through O(0,0) and $L\left(3,\frac{3}{2}\right)$ is $3y = \frac{3}{2}x \Rightarrow y = \frac{1}{2}x$ ....(1)

The equation of line OL' passing through O(0,0) and $L'\left(-3,\frac{3}{2}\right)$ is $-3y=\frac{3}{2}x \Rightarrow y=-\left(\frac{1}{2}\right)x$.......(2)

Let $m_1 = \frac{1}{2}, m_2 = -\frac{1}{2}$ be the slopes of the lines (1) & (2).

If θ be the angle between (1)&(2) then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$$= \left|\frac{\frac{1}{2}+\frac{1}{2}}{1+\frac{1}{2}\left(-\frac{1}{2}\right)}\right| = \frac{1}{\frac{3}{4}} = \frac{4}{3} \Rightarrow \tan\theta = \frac{4}{3} \Rightarrow \theta = \operatorname{Tan}^{-1}\left(\frac{4}{3}\right)$$

---

5 **If Q is the foot of the perpendicular from a point P on the parabola $y^2 = 8(x – 3)$ to its directrix. S is the focus of the parabola and if SPQ is an equilateral triangle then find the length of side of the triangle.**

EAM Q

Sol: Given parabola is $y^2 = 8(x – 3)$

Comparing with $(y–k)^2 = 4a(x–h)$ we get $4a = 8 \Rightarrow a = 2$

Vertex A(h, k) = (3,0) and the focus S (h+a,k) = (3+2, 0)= (5, 0)

But SPQ is equilateral triangle $\Rightarrow \angle SQP = 60^\circ$ and $\angle SQZ = 30^\circ$

From ΔSZQ we have $\sin 30^\circ = \frac{SZ}{SQ} \Rightarrow \frac{1}{2} = \frac{SZ}{SQ} \Rightarrow$ SQ=2(SZ)=2(4)=8

Hence length of each side of the triangle is 8 units.

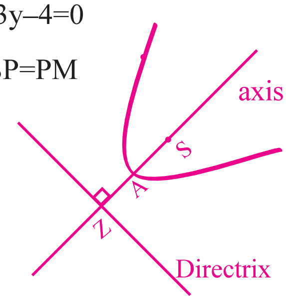
6 **Find the equation of the parabola whose focus is (–2,3) and directrix is the line 2x+3y–4=0. Also find the length of the latusrectum and the equation of the axis of the parabola.**

EAM Q

Sol: Let $P(x_1,y_1)$ be a point on the parabola

Given that focus S=(–2, 3) and the equation of the directrix is 2x+3y–4=0

Now, using the focus directrix property of the parabola we have SP=PM



$$\Rightarrow \sqrt{(x_1+2)^2+(y_1-3)^2} = \frac{|2x_1+3y_1-4|}{\sqrt{4+9}}$$

$$\Rightarrow 13((x_1+2)^2+(y_1-3)^2) = (2x_1+3y_1-4)^2$$

$$\Rightarrow 13(x_1^2+4x_1+4+y_1^2-6y_1+9) = (2x_1+3y_1-4)^2$$

$$\Rightarrow 13(x_1^2+y_1^2+4x_1-6y_1+13) = (2x_1+3y_1-4)^2$$

$$\Rightarrow 13x_1^2+13y_1^2+52x_1-78y_1+169 = 4x_1^2+9y_1^2+16+12x_1y_1-16x_1-24y_1$$

$$\Rightarrow 9x_1^2-12x_1y_1+4y_1^2+68x_1-54y_1+153=0$$

$\therefore$ Equation of the parabola is $9x^2-12xy+4y^2+68x-54y+153=0$

Perpendicular distance from S(–2,3) to the directrix 2x+3y–4=0 is

$$2a = \frac{|2(-2)+3(3)-4|}{\sqrt{4+9}} = \frac{1}{\sqrt{13}}$$

$\therefore$ Length of the latus rectum = $4a = 2(2a) = 2\left(\frac{1}{\sqrt{13}}\right) = \frac{2}{\sqrt{13}}$

We know that the axis is perpendicular to the directrix, and it passes through the focus.

Equation of the directrix is 2x+3y–4=0 .

Hence equation of the axis is taken as 3x–2y+k=0

If this line passes through S(–2,3) then –6–6+k=0 $\Rightarrow$ k=12

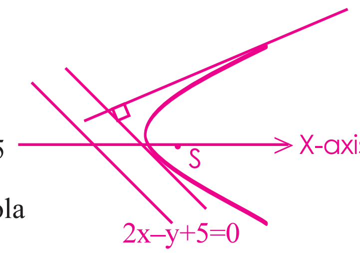
$\therefore$ Equation of the axis is 3x–2y+12=0

7 **Find the equations of tangents to the parabola $y^2 = 16x$ which are parallel and perpendicular respectively to the line 2x–y+5 =0. Find the coordinates of their points of contact also.**

Sol: The given parabola is $y^2 = 16x \Rightarrow 4a = 16 \Rightarrow a = 4$

The given line $2x-y+5=0 \Rightarrow y = 2x+5$

Comparing this equation with $y = mx+c$, we get $m=2$, $c=5$



(i) The equation of the tangent with slope m to the parabola

$y^2 = 4ax$ is $y = mx + \frac{a}{m} \Rightarrow y = 2x + \frac{4}{2} \Rightarrow y = 2x + 2$

Also the point of contact $= \left(\frac{a}{m^2}, \frac{2a}{m}\right) = \left(\frac{4}{2^2}, \frac{2\times 4}{2}\right) = \left(\frac{4}{4}, 4\right) = (1,4)$

(ii) As the slope of the tangent is m =2, the slope of its perpendicular is m= –1/2

$\therefore$ the equation of the perpendicular tangent with slope m = –1/2 to $y^2 = 16x$ is

$y = mx + \frac{a}{m} \Rightarrow y = -\frac{1}{2}x - \frac{4}{1/2} \Rightarrow y = -\frac{x}{2} - 8 \Rightarrow 2y = -x - 16 \Rightarrow x + 2y + 16 = 0$

Also the point of contact $= \left(\frac{a}{m^2}, \frac{2a}{m}\right) = \left(\frac{4}{(-1/2)^2}, \frac{2\times 4}{-1/2}\right) = \left(\frac{4}{1/4}, -\frac{8}{1/2}\right) = (16,-16)$

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8 **Prove that the normal chord at the point other than origin, whose ordinate is equal to its abscissa subtends a right angle at the focus.**

Sol: Let the equation of the parabola be $y^2=4ax$ ......(1)

$P(at^2, 2at)$ be any point on the parabola for which the abscissa is equal to the ordinate.

$\Rightarrow at^2=2at \Rightarrow t=0$ (or) $t=2$ but $t\neq 0$.

For t=2 we get the point (4a,4a) at which the equation of the normal is $y+2x=2a(2)+a(2)^3$.

$\Rightarrow y=12a-2x$ ....(2)

Substituting the value of $y=12a-2x$ in $y^2=4ax$ we get $(12a-2x)^2=4ax$

$\Rightarrow x^2-13ax+36a^2=(x-4a)(x-9a)=0 \Rightarrow x=4a, 9a$

Now, $x=4a \Rightarrow y=4a$ and $x=9a \Rightarrow y=-6a$

$\therefore$ the other point of intersection of the normal at P(4a, 4a) on the parabola is Q(9a, −6a).

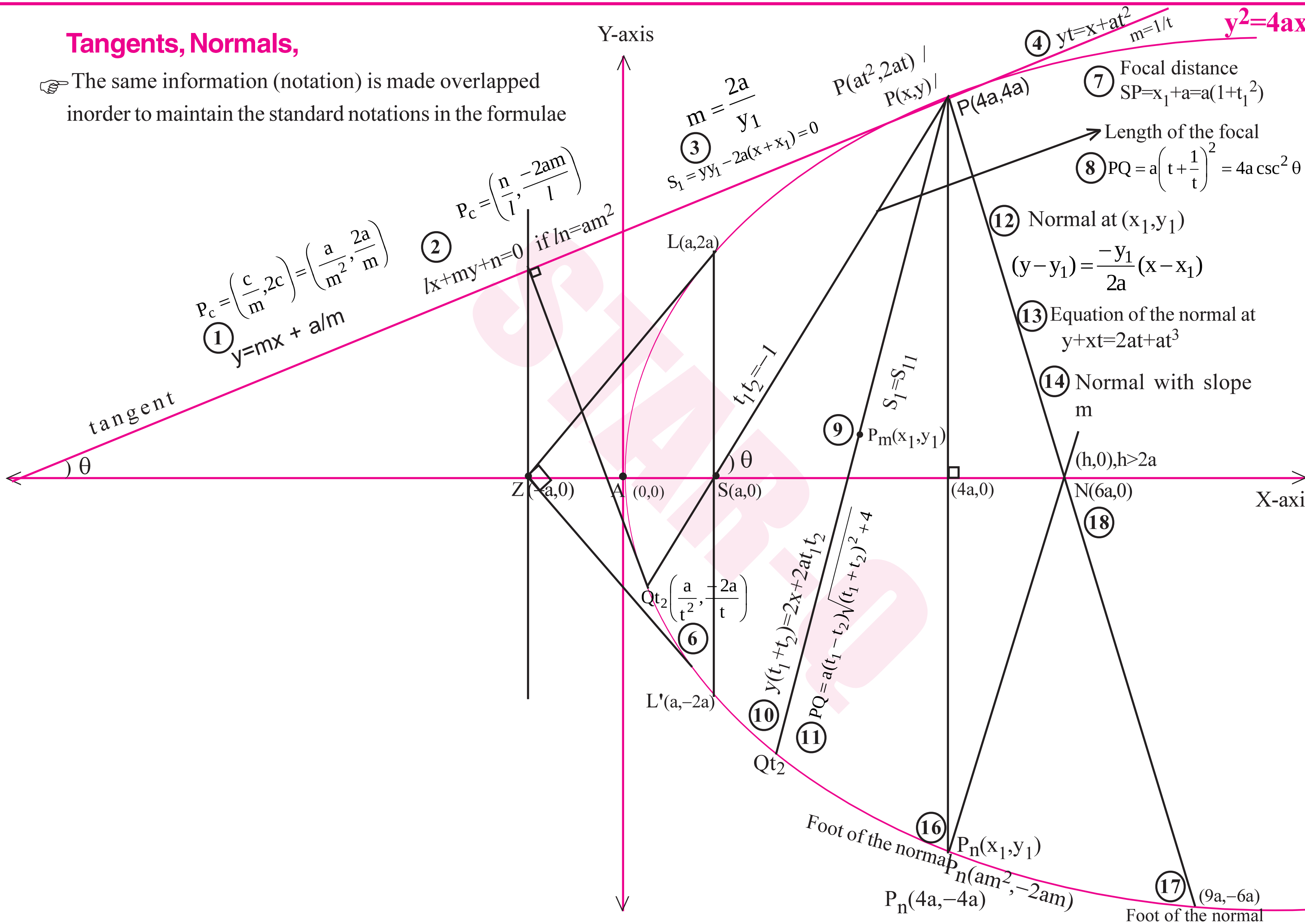
Also, we have S(a,0)

$\Rightarrow$ Slope of the line $\overleftrightarrow{SP} = m_1 = \frac{4a-0}{4a-a} = \frac{4}{3}$; Slope of the line $\overleftrightarrow{SQ} = m_2 = \frac{-6a-0}{9a-a} = -\frac{3}{4}$

Now, $m_1m_2 = \left(\frac{4}{3}\right)\left(-\frac{3}{4}\right) = -1 \Rightarrow \overleftrightarrow{SP}$ and $\overleftrightarrow{SQ}$ are at right angles $\Rightarrow$ PQ subtends a right angle at S

Tangents, Normals,

☞ The same information (notation) is made overlapped inorder to maintain the standard notations in the formulae



The Parametric stuff