WELCOME

STAR 'QR CODE'

DIGITAL MATERIAL

MATRICES -INDEX

3.MATRICES

1. INTRODUCTION PAGE

Sections	No. of periods (24)	Weightage in IPE [2x2+1x4 +2x7=22]
1. Basic Terminology, Types, Addition, Scalar Multiplication of Matrices	3	2 or 4 Marks
2. Multiplication of Matrices	4	4 Marks
3. Transpose, Determinant, Adjoint, Inverse of a matrix.	4	4 or 7 Marks
4. Properties of Determinants	5	7 Marks
5. Rank, Elementary Transformations	2	2 Marks
6. System of simultaneous equations	6	7 Marks

The name "Matrix" to a rectangular arrangement of numbers is given by J.J.Sylvester (1814 - 1897). Arthur Cayley (1821-1925) represented the system of the equations in the matrix equation form. In 1925 famous Physicist Heisenburg applied the theory of matrices in Atomic Structure of Quantum mechanics. Today, the theory of matrices is used as an indispensable tool in the study of Physical science, Engineering, Statistics, Economics, Sociology, computer game graphics etc.,

Matrix multiplication exhibits few different properties than that of the multiplication of real numbers. Eg: (i) Multiplication of matrices does not satisfy the commutative law i.e., $AB^{1}BA$. (ii) we can find zero divisors in matrices i.e., we can find $A^{1}0$, $B^{1}0$ such that $AB=0$. (iii) Cancellation law is not valid i.e., $AB=AC$, $A^{1}0$ then we can find matrices B, C such that $B^{1}C$.

In this first section, we recapitulate the basic concepts and extend them for further treatment. Types of matrices, Properties and Theorems regarding basic operations on matrices like addition, Scalar multiplication, Matrix multiplication are discussed. Also transpose, determinant, adjoint, inverse of a matrix and their properties are discussed.

Properties of determinants are helpful to handle some hardnut determinants easily. The theory of matrices is used to study the existence and nature of solutions of system of simultaneous equations. The concept of Rank of a matrix facilities the process of consistency and inconsistency of a given system of equations.

2. Proofs of Results of Properties of Determinants

1. Theorem 1: If A is a square matrix, then det A=det A^T i.e. $|A|=|A^T|$

Proof: Let $A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \Rightarrow A^T=\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ Now det A$=a_1A_1+a_2A_2+a_3A_3=$det A^T.

Ex: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$

2. Theorem 2: The sign of the determinant of a square matrix changes if any two rows (or columns) in the matrix are interchanged.

Proof: Let $A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, B=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \end{bmatrix}$

$$\det A = a_1(b_2c_3-b_3c_2)-b_1(a_2c_3-a_3c_2)+c_1(a_2b_3-a_3b_2)$$
$$= a_1b_2c_3-a_1b_3c_2-a_2b_1c_3+a_3b_1c_2+a_2b_3c_1-a_3b_2c_1$$
$$\det B = a_1(b_3c_2-b_2c_3)-b_1(a_3c_2-a_2c_3)+c_1(a_3b_2-a_2b_3)$$
$$= a_1b_3c_2-a_1b_2c_3-a_1b_2c_3+a_2b_1c_3+a_3b_2c_1-a_2b_3c_1=-\det A$$

Ex: $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}=(1x4)-(3x2)=-2$ and $\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}=(3x2)-(1x4)=2$

3. Theorem 3: If any two rows (or columns) of a squre matrix are identical, the value of the determinant of the matrix is zero.

Proof: Let $A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$

$$\det A = a_1(b_1c_2-b_2c_1)-b_1(a_1c_2-a_2c_1)+c_1(a_1b_2-a_2b_1)$$
$$= a_1b_1c_2-a_1b_2c_1-a_1b_1c_2+a_2b_1c_1+a_1b_2c_1-a_2b_1c_1=0$$

Ex: $A=\begin{bmatrix} a & d & a \\ b & e & b \\ c & f & c \end{bmatrix} \Rightarrow |A|=0$

4. Theorem 4: If all the elements of a row (or column) of a square matrix are multiplied by a number k then the value of the determinant of the matrix obtained is k times the determinant of the given matrix. (For a 3 x 3 matrix A, $|kA|=k^3|A|$)

Proof: Let $A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, B=\begin{bmatrix} ka_1 & b_1 & c_1 \\ ka_2 & b_2 & c_2 \\ ka_3 & b_3 & c_3 \end{bmatrix}$

$$\det A = a_1(b_2c_3-b_3c_2)-b_1(a_2c_3-a_3c_2)+c_1(a_2b_3-a_3b_2)$$
$$= a_1b_2c_3-a_1b_3c_2-a_2b_1c_3+a_3b_1c_2+a_2b_3c_1-a_3b_2c_1$$
$$\det B = ka_1(b_2c_3-b_3c_2)-b_1(ka_2c_3-ka_3c_2)+c_1(ka_2b_3-ka_3b_2)$$
$$= ka_1b_2c_3-ka_1b_3c_2-ka_2b_1c_2+ka_3b_1c_2+ka_2b_3c_1-ka_3b_2c_1=k\det A$$

Ex: If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = p$ then $\begin{vmatrix} ka & kb & kc \\ d & e & f \\ g & h & i \end{vmatrix} = kp, \begin{vmatrix} ka & kb & kc \\ ld & le & lj \\ g & h & i \end{vmatrix} = (kl)p, \begin{vmatrix} ka & kb & kc \\ ld & le & lj \\ mg & mh & mi \end{vmatrix} = (klm)p$

Also $k\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} ka & kb & kc \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & kb & c \\ d & ke & f \\ g & kh & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ kg & kh & ki \end{vmatrix}$ etc.,

☞ $k\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} ka & kb & kc \\ kd & ke & kf \\ kg & kh & ki \end{bmatrix}$ & $\begin{vmatrix} ka & kb & kc \\ kd & ke & kf \\ kg & kh & ki \end{vmatrix} = k^3\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

5. Theorem 5: If each element in a row (or column) of a square matrix is the sum of the two numbers, then its determinant can be expressed as the sum of the determinants of two square matrices.

Proof: Let $A = \begin{bmatrix} a_1+x_1 & b_1 & c_1 \\ a_2+x_2 & b_2 & c_2 \\ a_3+x_3 & b_3 & c_3 \end{bmatrix}, B = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, C = \begin{bmatrix} x_1 & b_1 & c_1 \\ x_2 & b_2 & c_2 \\ x_3 & b_3 & c_3 \end{bmatrix}$

$$\det A = (a_1+x_1)(b_2c_3-b_3c_2)-(a_2+x_2)(b_1c_3-b_3c_1)+(a_3+x_3)(b_1c_2-b_2c_1)$$
$$= a_1(b_2c_3-b_3c_2)+x_1(b_2c_3-b_3c_2)-a_2(b_1c_3-b_3c_1)-x_2(b_1c_3-b_3c_1)$$
$$+a_3(b_1c_2-b_2c_1)+x_3(b_1c_2-b_2c_1)$$
$$= a_1(b_2c_3-b_3c_2)-a_2(b_1c_3-b_3c_1)+a_3(b_1c_2-b_2c_1)+x_1(b_2c_3-b_3c_2)$$
$$-x_2(b_1c_3-b_3c_1)+x_3(b_1c_2-b_2c_1) = \det B + \det C$$

Ex: $\begin{vmatrix} a_1 & b_1+x & c_1 \\ a_2 & b_2+y & c_2 \\ a_3 & b_3+z & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & x & c_1 \\ a_2 & y & c_2 \\ a_3 & z & c_3 \end{vmatrix}$

6.Theorem 6: If the elements of a row (or column) of a square matrix are added with k times the corresponding elements of another row (or column), then the value of the determinant of the matrix obtained is same as the value of determinant of the given matrix.

Proof: Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, B = \begin{bmatrix} a_1+kc_1 & b_1 & c_1 \\ a_2+kc_2 & b_2 & c_2 \\ a_3+kc_3 & b_3 & c_3 \end{bmatrix}$

$$\det B = \begin{vmatrix} a_1+kc_1 & b_1 & c_1 \\ a_2+kc_2 & b_2 & c_2 \\ a_3+kc_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} kc_1 & b_1 & c_1 \\ kc_2 & b_2 & c_2 \\ kc_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + k\begin{vmatrix} c_1 & b_1 & c_1 \\ c_2 & b_2 & c_2 \\ c_3 & b_3 & c_3 \end{vmatrix}$$

$= \det A + k0 = \det A$

Ex: $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = \begin{vmatrix} x_1+ky_1 & y_1 & z_1 \\ x_2+ky_2 & y_2 & z_2 \\ x_3+ky_3 & y_3 & z_3 \end{vmatrix}$

☞ The value of the determinant of a matrix do not change with the operations like R_1+R_2, R_1-R_2, R_1+kR_2, $R_1+R_2+R_3$ or C_1+C_2, C_1-C_2, C_1+kC_2, $C_1+C_2+C_3$

7. **Theorem 7: The sum of the products of the elements of any row (or column) of a square matrix with the cofactors of the corresponding elements of another row (or column) of the matrix is zero.**

Proof: Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

$$a_1A_2 + b_1B_2 + c_1C_2 = -a_1\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_1\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - c_1\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$$

$$= -a_1(b_1c_3 - b_3c_1) + b_1(a_1c_3 - a_3c_1) - c_1(a_1b_3 - a_3b_1)$$

$$= -a_1b_1c_3 + a_1b_3c_1 + a_1b_1c_3 - a_3b_1c_1 - a_1b_3c_1 + a_3b_1c_1 = 0$$

8. If the elements of a row/column of a square matrix are proportional to some other row i.e.,k times the elements of another row/column, then the determinant of the matrix is zero.

Ex: $\begin{vmatrix} a_1 & kb_1 & b_1 \\ a_2 & kb_2 & b_2 \\ a_3 & kb_3 & b_3 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 & b_1 \\ a_2 & b_2 & b_2 \\ a_3 & b_3 & b_3 \end{vmatrix} = 0$

9. **Theorem 8: If the elements of a square matrix are polynomials in x and two rows (or columns) are identical when x=a, then (x–a) is a factor of the determinant of the matrix.**

Proof: Let $F(x) = \begin{bmatrix} f_1(x) & g_1(x) & h_1(x) \\ f_2(x) & g_2(x) & h_2(x) \\ f_3(x) & g_3(x) & h_3(x) \end{bmatrix}$

$$F(a) = \begin{vmatrix} f_1(a) & g_1(a) & h_1(a) \\ f_2(a) & g_2(a) & h_2(a) \\ f_3(a) & g_3(a) & h_3(a) \end{vmatrix} = 0$$

∴ By factor theorem (x–a) is a factor of F(x)

10. **The determinant of triangular matrix is the product of the diagonal elements of the matrix** $\begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & b_2 & b_3 \\ 0 & 0 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & 0 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3$

11. **Theorem 9:If A and B are two square matrices of same type, then det (AB)=det A.det B.**

Proof: Consider the matrices $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

Then, $\det A = a_{11}a_{22} - a_{21}a_{12}$;

$\det B = b_{11}b_{22} - b_{21}b_{12}$.

Now $AB = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{bmatrix}$

$$\det(AB) = (a_{11}b_{11}+a_{12}b_{21})(a_{21}b_{12}+a_{22}b_{22})-(a_{21}b_{11}+a_{22}b_{21})(a_{11}b_{12}+a_{12}b_{22})$$

$$= a_{11}a_{21}b_{11}b_{12}+a_{11}a_{22}b_{11}b_{22}+a_{12}a_{21}b_{12}b_{21}+a_{12}a_{22}b_{21}b_{22}$$

$$-a_{11}a_{21}b_{11}b_{12}-a_{12}a_{21}b_{11}b_{22}-a_{11}a_{22}b_{12}b_{21}-a_{12}a_{22}b_{12}b_{22}$$

$$= a_{11}a_{22}b_{11}b_{22}+a_{12}a_{21}b_{12}b_{21}-a_{12}a_{21}b_{11}b_{22}-a_{11}a_{22}b_{12}b_{21}$$

$$= a_{11}a_{22}(b_{11}b_{22}-b_{12}b_{21})-a_{12}a_{21}(b_{11}b_{22}-b_{12}b_{21})$$

$$= (a_{11}a_{22}-a_{12}a_{21})(b_{11}b_{22}-b_{12}b_{21}) = (\det A)(\det B)$$

If A and B are matrices of order three then also in a similar manner we can show that det(AB)=(det A)(det B)

Notation: While evaluating determinants, we use the following notation.

(i) $R_1 \rightarrow kR_1$ means that the elements of R_1 are multiplied by k.

(ii) $R_1 \rightarrow R_1+kR_2$ means that the elements of R_1 are added with k times the corresponding elements of R_2.

☞ The following observations are helpful to avoid the confusion between the matrix representation and determinant respresentation.

(i) matrix is an array of numbers but determinant of a matrix is a real value

(ii) In matrices we deal with rectangular ones, but in determinants we deal with the determinants of square matrices only.

(iii) Distinction should be made clearly especially w.r.to property 4 of determinants.

(iv) The process of matrix multiplication is unique but the process of determianant multiplication can be done in several ways.

MATH BEATS!

PECULIAR PROPERTIES of Matrix Multiplication

In Real Numbers	In Matrices
1) ab=ba (2×3 = 3×2)	1) AB≠BA (Generally); Rarely AB=BA
2) ab=0 ⇒ a=0 (or) b=0 **Ex:** 2x=0 ⇒ x=0	2) AB=O need not imply A=O (or) B=O **Ex:** $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
3) ab=ac, a≠0 ⇒ b=c **Ex:** $\not{2}x = \not{2}y \Rightarrow x = y$	3) AB=AC, A≠O ⇒ B need not = C **Ex:** $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}$
4) ab=b then a=1, b≠0 **Ex:** 2x=2 ⇒ x=1	4) AB=B then A need not be the unit matrix I. **Ex:** $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

మొత్తానికి Matrix Multiplication కొంచెం తేడాగానే కనిపిస్తుంది...... కొంచెం కాదు బాగా....

3. ADDITIONAL QS WITH SOLUTIONS ON DETERMINANTS

1. If $\Delta_1 = \begin{vmatrix} a_1^2+b_1+c_1 & a_1a_2+b_2+c_2 & a_1a_3+b_3+c_3 \\ b_1b_2+c_1 & b_2^2+c_2 & b_2b_3+c_3 \\ c_3c_1 & c_3c_2 & c_3^2 \end{vmatrix}, \Delta_2 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ **then find the value of** $\frac{\Delta_1}{\Delta_2}$

Sol : $$\Delta_1 = \begin{vmatrix} a_1^2+b_1+c_1 & a_1a_2+b_2+c_2 & a_1a_3+b_3+c_3 \\ b_1b_2+c_1 & b_2^2+c_2 & b_2b_3+c_3 \\ c_3c_1 & c_3c_2 & c_3^2 \end{vmatrix} = c_3\begin{vmatrix} a_1^2+b_1+c_1 & a_1a_2+b_2+c_2 & a_1a_3+b_3+c_3 \\ b_1b_2+c_1 & b_2^2+c_2 & b_2b_3+c_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

On applying $R_2 \to R_2 - R_3$, we have

$$\Delta_1 = c_3\begin{vmatrix} a_1^2+b_1+c_1 & a_1a_2+b_2+c_2 & a_1a_3+b_3+c_3 \\ b_1b_2 & b_2^2 & b_2b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = c_3b_2\begin{vmatrix} a_1^2+b_1+c_1 & a_1a_2+b_2+c_2 & a_1a_3+b_3+c_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

On applying $R_1 \to R_1-(R_2+R_3)$, we have

$$\Delta_1 = c_3b_2\begin{vmatrix} a_1^2 & a_1a_2 & a_1a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = c_3b_2a_1\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1b_2c_3\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} [\because |A| = |A'|]$$

$$\therefore \Delta_1 = a_1b_2c_3 \times \Delta_2$$

$$\Rightarrow \frac{\Delta_1}{\Delta_2} = a_1b_2c_3$$

2. If $\Delta_1 = \begin{vmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos\gamma \\ \cos\beta & \cos\gamma & 1 \end{vmatrix}, \Delta_2 = \begin{vmatrix} 0 & \cos\alpha & \cos\beta \\ \cos\alpha & 0 & \cos\gamma \\ \cos\beta & \cos\gamma & 0 \end{vmatrix}$ and $\Delta_1=\Delta_2$, then show that $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

Sol : $\Delta_1 = \begin{vmatrix} 1 & \cos\alpha & \cos\beta \\ \cos\alpha & 1 & \cos\gamma \\ \cos\beta & \cos\gamma & 1 \end{vmatrix}$

$= 1(1-\cos^2\gamma)-\cos\alpha(\cos\alpha-\cos\beta\cos\gamma)+\cos\beta(\cos\alpha\cos\gamma-\cos\beta)$

$= 1-\cos^2\gamma-\cos^2\alpha+\cos\alpha\cos\beta\cos\gamma+\cos\alpha\cos\beta\cos\gamma-\cos^2\beta$

$= 1-\cos^2\gamma-\cos^2\alpha-\cos^2\beta+2\cos\alpha\cos\beta\cos\gamma$

Now, $\Delta_2= 0(0-\cos\gamma\cos\gamma)-\cos\alpha(0-\cos\beta\cos\gamma)+\cos\beta(\cos\alpha\cos\gamma-0)=2\cos\alpha\cos\beta\cos\gamma$

Given that $\Delta_1=\Delta_2$

$\Rightarrow 1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma = 2\cos\alpha\cos\beta\cos\gamma$

$\Rightarrow 1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma=0 \Rightarrow 1=\cos^2\alpha+\cos^2\beta+\cos^2\gamma$

---

3. If $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, B = \frac{1}{2}\begin{bmatrix} b+c & c-a & b-a \\ c-b & c+a & a-b \\ b-c & a-c & a+b \end{bmatrix}$ then S.T ABA^{-1} is a diagnoal matrix

Sol: $\det A = 0(-1)+1(1)+1(1) = 1+1=2$

The cofactor matrix of $A = \begin{bmatrix} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

$\Rightarrow \text{Adj } A=\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{\det A}(\text{Adj}A) = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

$$\text{Now, } AB=\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}.\frac{1}{2}\begin{bmatrix}b+c&c-a&b-a\\c-b&c+a&a-b\\b-c&a-c&a+b\end{bmatrix}=\frac{1}{2}\begin{bmatrix}c-b+b-c&c+a+a-c&a-b+a+b\\b+c+b-c&c-a+a-c&b-a+a+b\\b+c+c-b&c-a+c+a&b-a+a-b\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}0&2a&2a\\2b&0&2b\\2c&2c&0\end{bmatrix}=\begin{bmatrix}0&a&a\\b&0&b\\c&c&0\end{bmatrix}$$

$$ABA^{-1}=(AB)A^{-1}=\begin{bmatrix}0&a&a\\b&0&b\\c&c&0\end{bmatrix}.\frac{1}{2}\begin{bmatrix}-1&1&1\\1&-1&1\\1&1&-1\end{bmatrix}=\frac{1}{2}\begin{bmatrix}0&a&a\\b&0&b\\c&c&0\end{bmatrix}\begin{bmatrix}-1&1&1\\1&-1&1\\1&1&-1\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}0+a+a&0-a+a&0+a-a\\-b+0+b&b+0+b&b+0-b\\-c+c+0&c-c+0&c+c-0\end{bmatrix}=\frac{1}{2}\begin{bmatrix}2a&0&0\\0&2b&0\\0&0&2c\end{bmatrix}=\begin{bmatrix}a&0&0\\0&b&0\\0&0&c\end{bmatrix}$$

$\therefore ABA^{-1}$ is a diagonal matrix

---

4. Show that $\begin{vmatrix}-2a&a+b&c+a\\a+b&-2b&b+c\\c+a&c+b&-2c\end{vmatrix}=4(a+b)(b+c)(c+a)$

Sol: $\text{L.H.S}=\begin{vmatrix}-2a&a+b&c+a\\a+b&-2b&b+c\\c+a&c+b&-2c\end{vmatrix}$

$= -2a[4bc-(b+c)^2]-(a+b)[-2c(a+b)-(a+c)(b+c)]+(c+a)[(a+b)(b+c)+2b(a+c)]$

$= 2a[(b+c)^2-4bc]+(a+b)[2c(a+b)+(a+c)(b+c)]+(c+a)[(a+b)(b+c)+2b(a+c)]$

$= 2a[(b^2+c^2+2bc)-4bc]+(a+b)[2ca+2cb+ab+ac+cb+c^2]+(c+a)[ab+ac+b^2+bc+2ba+2bc]$

$= 2a[b^2+c^2-2bc]+(a+b)[3ca+3cb+ab+c^2]+(c+a)[3ab+ac+b^2+3bc]$

$= [2ab^2+2ac^2-4abc]+[3ca^2+3cba+a^2b+ac^2+3cab+3cb^2+ab^2+bc^2]$

$+[3abc+ac^2+b^2c+3bc^2+3a^2b+a^2c+ab^2+3abc]$

$= 4ab^2+4ac^2+8abc+4ca^2+4a^2b+4cb^2+4bc^2$

$= 4[ab^2+ac^2+ca^2+a^2b+cb^2+bc^2+2abc]$..........(1)

R.H.S $= 4(a+b)(b+c)(c+a)=4(a+b)[bc+ab+c^2+ac]=4[abc+a^2b+ac^2+a^2c+b^2c+b^2a+bc^2+abc]$

$=4[a^2b+ac^2+a^2c+b^2c+b^2a+bc^2+2abc]$.........(2)

From (1) & (2), L.H.S= R.H.S

5. If AB=I or BA=I. Then prove that 'A' is invertible and $B=A^{-1}$.

Sol: Given $AB=I \Rightarrow |AB|=|I| \Rightarrow |A||B|=I \Rightarrow |A|\neq 0 \Rightarrow$ A is a non-singular matrix

Again, $BA=I \Rightarrow |BA|=|I| \Rightarrow |B||A|=I \Rightarrow |A|\neq 0 \Rightarrow$ A is a non-singular matrix

Also, $AB=I$ or $BA=I \Rightarrow$ A is invertible $\quad \therefore A^{-1}$ exists.

Now, $AB=I \Rightarrow A^{-1}(AB)=A^{-1}(I) \Rightarrow (A^{-1}A)B=A^{-1} \Rightarrow IB=A^{-1} \Rightarrow B=A^{-1}$

---

6. For any square matrix A, show that AA' is symmetric.

Sol : To show that AA' is Symmetric, we have to show that (AA')'=AA'

L.H.S=(AA')'=(A')'A' [since (AB)' = B'A']

=AA' [since (A')'=A]

= R.H.S

Thus (AA')'=AA'

Hence, AA' is a symmetric matrix.

---

7. If $A=\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}, B=\frac{1}{2}\begin{bmatrix}b+c&c-a&b-a\\c-b&c+a&a-b\\b-c&a-c&a+b\end{bmatrix}$ then S.T. ABA^{-1} is a diagonal matrix

Sol : $\det A = 0(-1)+1(1)+1(1) = 1+1=2$

The cofactor matrix of $A=\begin{bmatrix}\begin{vmatrix}0&1\\1&0\end{vmatrix} & -\begin{vmatrix}1&1\\1&0\end{vmatrix} & \begin{vmatrix}1&0\\1&1\end{vmatrix}\\ -\begin{vmatrix}1&1\\1&0\end{vmatrix} & \begin{vmatrix}0&1\\1&0\end{vmatrix} & -\begin{vmatrix}0&1\\1&1\end{vmatrix}\\ \begin{vmatrix}1&1\\0&1\end{vmatrix} & -\begin{vmatrix}0&1\\1&1\end{vmatrix} & \begin{vmatrix}0&1\\1&0\end{vmatrix}\end{bmatrix}=\begin{bmatrix}-1&1&1\\1&-1&1\\1&1&-1\end{bmatrix}$

$\Rightarrow \text{Adj } A=\begin{bmatrix}-1&1&1\\1&-1&1\\1&1&-1\end{bmatrix} \therefore A^{-1}=\frac{1}{\det A}(\text{Adj}A)=\frac{1}{2}\begin{bmatrix}-1&1&1\\1&-1&1\\1&1&-1\end{bmatrix}$

Now, $AB=\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}\frac{1}{2}\begin{bmatrix}b+c&c-a&b-a\\c-b&c+a&a-b\\b-c&a-c&a+b\end{bmatrix}=\frac{1}{2}\begin{bmatrix}c-b+b-c&c+a+a-c&a-b+a+b\\b+c+b-c&c-a+a-c&b-a+a+b\\b+c+c-b&c-a+c+a&b-a+a-b\end{bmatrix}$

$=\frac{1}{2}\begin{bmatrix}0&2a&2a\\2b&0&2b\\2c&2c&0\end{bmatrix}=\begin{bmatrix}0&a&a\\b&0&b\\c&c&0\end{bmatrix}$

$$ABA^{-1} = (AB)A^{-1} = \begin{bmatrix} 0 & a & a \\ b & 0 & b \\ c & c & 0 \end{bmatrix} \frac{1}{2} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & a & a \\ b & 0 & b \\ c & c & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 0+a+a & 0-a+a & 0+a-a \\ -b+0+b & b+0+b & b+0-b \\ -c+c+0 & c-c+0 & c+c-0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2a & 0 & 0 \\ 0 & 2b & 0 \\ 0 & 0 & 2c \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

$\therefore$ ABA^{-1} is a diagonal matrix

---

8. A trust fund has to invest Rs.30,000 in two different types of bonds. The first bond pays 5% interest per year, and the second bond pays 7% interest per year. Using matrix multiplication, determine how to divide Rs.30,000 among the two types of bonds, if the trust fund must obtain an annual total interest of (a) Rs.1800 (b) Rs.2000.

Sol : If the amount in the first bond is 'x' then the amount in the second bond will be 30,000–x . Rates of interest are 5% and 7% respectively.

(a) $[x \quad 30,000-x]\begin{bmatrix} 5\% \\ 7\% \end{bmatrix} = [1800] \Rightarrow \frac{5}{100}x + \frac{7}{100}(30,000-x) = 1800$

$\Rightarrow 5x + 2,10,000 - 7x = 1,80,000 \Rightarrow -2x = 1,80,000 - 2,10,000 = -30,000 \Rightarrow x = 15,000$

$\therefore$ Amount in the first bond = 15,000

Hence amount in the second bond = 30,000 – 15,000 = 15,000

(b) $[x \quad 30,000-x]\begin{bmatrix} 5\% \\ 7\% \end{bmatrix} = [2000] \Rightarrow \frac{5}{100}x + \frac{7}{100}(30,000-x) = 2000$

$\Rightarrow 5x + 2,10,000 - 7x = 2,00,000 \Rightarrow -2x = 2,00,000 - 2,10,000 = -10,000 \Rightarrow x = 5,000$

$\therefore$ Amount in the first bond = Rs. 5,000

Amount in the second bond = 30,000 – 5,000 = Rs.25,000

4. ADDITIONAL QS WITH SOLUTIONS ON SYSTEM OF EQUATIONS

1. Apply the test of rank to examine whether the following equations are consistent or not 2x−y+3z=8, −x+2y+z=4, 3x+y−4z=0 and if consistent, find the complete solution.

Sol: The augmented matrix of the given system of equations, is

$$[A\ D]=\begin{bmatrix}2 & -1 & 3 & 8\\ -1 & 2 & 1 & 4\\ 3 & 1 & -4 & 0\end{bmatrix}\sim\begin{bmatrix}-1 & 2 & 1 & 4\\ 2 & -1 & 3 & 8\\ 3 & 1 & -4 & 0\end{bmatrix}\text{(on interchanging } R_1 \text{ and } R_2)$$

We transform the above matrix into an upper triangular matrix.

$$\sim\begin{bmatrix}-1 & 2 & 1 & 4\\ 0 & 3 & 5 & 16\\ 0 & 7 & -1 & 12\end{bmatrix}\text{(on applying } R_2\to R_2+2R_1,\ R_3\to R_3+3R_1)$$

$$\sim\begin{bmatrix}-1 & 2 & 1 & 4\\ 0 & 3 & 5 & 16\\ 0 & 0 & -38 & -76\end{bmatrix}\text{(on applying } R_3\to 3R_3-7R_2)\ \ .....(I)$$

$$\text{Now det}\begin{bmatrix}-1 & 2 & 1\\ 0 & 3 & 5\\ 0 & 0 & -38\end{bmatrix}=(-1)(3)(-38)=114\neq 0$$

Hence rank(A)=rank[A D]=3

$\therefore$ the system has a unique solution.

From (I), the equivalent system of equations are $-x+2y+z=4;\ 3y+5z=16;\ -38z=-76$

$\Rightarrow z=2$ hence $3y+10=16\Rightarrow y=2$, hence $-x+4+2=4\Rightarrow x=2$

$\therefore$ x=2, y=2, z=2 is the solution.

2. Show that the following system of equations is consistent and solve it completely. x+y+z=3; 2x+2y−z=3; x+y−z=1

Sol: The augmented matrix of the given system of equations, is $[A\ D]=\begin{bmatrix}1 & 1 & 1 & 3\\ 2 & 2 & -1 & 3\\ 1 & 1 & -1 & 1\end{bmatrix}$

$\sim\begin{bmatrix}1 & 1 & 1 & 3\\ 0 & 0 & -3 & -3\\ 0 & 0 & -2 & -2\end{bmatrix}$ (On applying $R_2\to R_2-2R_1$, $R_3\to R_3-R_1$)

$\sim\begin{bmatrix}1 & 1 & 1 & 3\\ 0 & 0 & -3 & -3\\ 0 & 0 & 0 & 0\end{bmatrix}$ ....(I) On applying $R_3\to 3R_3-2R_2$ we get

Clearly all the submatrices of order 3 of the above matrix are singular.

Hence rank A ≠ 3, and rank [A D]≠3

Now the non-singular matrix $\begin{bmatrix}1 & 1\\ 0 & -3\end{bmatrix}$ is a submatrix of both A and [AD].

Hence rank(A)=rank [AD]=2.

Hence the system is consistent and it has infinitely many solutions.

From (I), the equivalent set of equations are

$x+y+z=3$

$-3z=-3$

Hence $z=1\Rightarrow x+y=2$.

For $k\in R$, let $x=k\Rightarrow y=2-k,\ z=1$ is the infinite solution set.

3. By using Cramer's solve x+y+z=1, 2x+2y+3z=6, x+4y+9z=3

Sol: Given equations can be written as AX = D, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, D = \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix}$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 1(18-12) - 1(18-3) + 1(8-2) = 6 - 15 + 6 = -3$$

$$\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ 6 & 2 & 3 \\ 3 & 4 & 9 \end{vmatrix} = 1(18-12) - 1(54-9) + 1(24-6) = 6 - 45 + 18 = -21$$

$$\Delta_2 = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 6 & 3 \\ 1 & 3 & 9 \end{vmatrix} = 1(54-9) - 1(18-3) + 1(6-6) = 45 - 15 = 30$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 6 \\ 1 & 4 & 3 \end{vmatrix} = 1(6-24) - 1(6-6) + 1(8-2) = -18 - 0 + 6 = -12$$

$\therefore$ By Carmer's rule $x = \frac{\Delta_1}{\Delta} = \frac{-21}{-3} = 7$, $y = \frac{\Delta_2}{\Delta} = \frac{30}{-3} = -10$ $z = \frac{\Delta_3}{\Delta} = \frac{-12}{-3} = 4$

$\therefore$ The solution is $x = 7, y = -10, z = 4$

4. By using Cramer's solve x+y+z=9, 2x+5y+7z=52, 2x+y–z=0

Sol: Given equations can be written as AX = D, where

$$A=\begin{bmatrix}1&1&1\\2&5&7\\2&1&-1\end{bmatrix}, X=\begin{bmatrix}x\\y\\z\end{bmatrix}, D=\begin{bmatrix}9\\52\\0\end{bmatrix}$$

$$\Delta=\begin{vmatrix}1&1&1\\2&5&7\\2&1&-1\end{vmatrix}=1(-5-7)-1(-2-14)+1(2-10)=-12+16-8=-4$$

$$\Delta_1=\begin{vmatrix}9&1&1\\52&5&7\\0&1&-1\end{vmatrix}=9(-5-7)-1(-52-0)+1(52-0)=-108+52+52=-4$$

$$\Delta_2=\begin{vmatrix}1&9&1\\2&52&7\\2&0&-1\end{vmatrix}=1(-52-0)-9(-2-14)+1(0-104)=-52+144-104=-12$$

$$\Delta_3=\begin{vmatrix}1&1&9\\2&5&52\\2&1&0\end{vmatrix}=1(0-52)-1(0-104)+9(2-10)=-52+104-72=-20$$

$\therefore$ By Carmer's rule $x=\frac{\Delta_1}{\Delta}=\frac{-4}{-4}=1,\ y=\frac{\Delta_2}{\Delta}=\frac{-12}{-4}=3,\ z=\frac{\Delta_3}{\Delta}=\frac{-20}{-4}=5$

$\therefore$ The solution is x = 1, y =3, z = 5

5. By using Matrix inversion method solve 2x–y+3z=9, x+y+z=6, x–y+z=2 .

Sol: The matrix equation corresponding to the given system of equations be AX=D, where

$$A=\begin{bmatrix}2&-1&3\\1&1&1\\1&-1&1\end{bmatrix};\ X=\begin{bmatrix}x\\y\\z\end{bmatrix},\ D=\begin{bmatrix}9\\6\\2\end{bmatrix}\qquad \therefore \text{ The solution of AX=D is } X=A^{-1}D$$

First we find A^{-1}

$$|A|=\begin{vmatrix}2&-1&3\\1&1&1\\1&-1&1\end{vmatrix}=2(1+1)+1(1-1)+3(-1-1)=2(2)+1(0)+3(-2)=4+0-6=-2\neq 0$$

The co-factor matrix of A is

$$\begin{bmatrix}+\begin{vmatrix}1&1\\-1&1\end{vmatrix}&-\begin{vmatrix}1&1\\1&1\end{vmatrix}&+\begin{vmatrix}1&1\\1&-1\end{vmatrix}\\-\begin{vmatrix}-1&3\\-1&1\end{vmatrix}&+\begin{vmatrix}2&3\\1&1\end{vmatrix}&-\begin{vmatrix}2&-1\\1&-1\end{vmatrix}\\+\begin{vmatrix}-1&3\\1&1\end{vmatrix}&-\begin{vmatrix}2&3\\1&1\end{vmatrix}&+\begin{vmatrix}2&-1\\1&1\end{vmatrix}\end{bmatrix}=\begin{bmatrix}(1+1)&-(1-1)&(-1-1)\\-(-1+3)&(2-3)&-(-2+1)\\(-1-3)&-(2-3)&(2+1)\end{bmatrix}=\begin{bmatrix}2&0&-2\\-2&-1&1\\-4&1&3\end{bmatrix}$$

$$\Rightarrow \text{Adj } A=\begin{bmatrix}2&-2&-4\\0&-1&1\\-2&1&3\end{bmatrix}$$

$$\therefore A^{-1}=\frac{1}{\det A}(\text{Adj}A)=\frac{1}{-2}\begin{bmatrix}2&-2&-4\\0&-1&1\\-2&1&3\end{bmatrix}$$

$$\text{Now, } X=A^{-1}D=\frac{1}{-2}\begin{bmatrix}2&-2&-4\\0&-1&1\\-2&1&3\end{bmatrix}\begin{bmatrix}9\\6\\2\end{bmatrix}=\frac{-1}{2}\begin{bmatrix}18-12-8\\0-6+2\\-18+6+6\end{bmatrix}=\frac{-1}{2}\begin{bmatrix}-2\\-4\\-6\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}$$

$$\therefore X=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}$$

$\therefore$ The solution is x=1, y=2, z=3

6. By using Matrix inversion method, solve x+y+z=9, 2x+5y+7z=52, 2x+y–z=0 .

Sol: The matrix equation corresponding to the given system of equations be AX=D, where

$$A=\begin{bmatrix}1&1&1\\2&5&7\\2&1&-1\end{bmatrix};\ X=\begin{bmatrix}x\\y\\z\end{bmatrix},\ D=\begin{bmatrix}9\\52\\0\end{bmatrix}\qquad \therefore \text{ The solution of AX=D is } X=A^{-1}D$$

First we find A^{-1}

$$\det A=|A|=\begin{vmatrix}1&1&1\\2&5&7\\2&1&1\end{vmatrix}=1(-5-7)-1(-2-14)+1(2-10)=1(-12)-1(-16)+1(-8)=-12+16-8=-4\neq 0$$

The co-factor matrix of A is

$$\begin{bmatrix}+\begin{vmatrix}5&7\\1&-1\end{vmatrix} & -\begin{vmatrix}2&7\\2&-1\end{vmatrix} & +\begin{vmatrix}2&5\\2&1\end{vmatrix}\\ -\begin{vmatrix}1&1\\1&-1\end{vmatrix} & +\begin{vmatrix}1&1\\2&-1\end{vmatrix} & -\begin{vmatrix}1&1\\2&1\end{vmatrix}\\ +\begin{vmatrix}1&1\\5&7\end{vmatrix} & -\begin{vmatrix}1&1\\2&7\end{vmatrix} & +\begin{vmatrix}1&1\\2&5\end{vmatrix}\end{bmatrix}=\begin{bmatrix}(-5-7)&-(2-14)&(2-10)\\-(-1-1)&(-1-2)&-(1-2)\\(7-5)&-(7-2)&(5-2)\end{bmatrix}=\begin{bmatrix}-12&16&-8\\2&-3&1\\2&-5&3\end{bmatrix}$$

$$\Rightarrow \text{Adj } A=\begin{bmatrix}-12&2&2\\16&-3&-5\\-8&1&3\end{bmatrix}$$

$$\therefore A^{-1}=\frac{1}{\det A}(\text{Adj}A)=\frac{1}{-4}\begin{bmatrix}-12&2&2\\16&-3&-5\\-8&1&3\end{bmatrix}$$

$$\text{Now, } X=A^{-1}D=\frac{1}{-4}\begin{bmatrix}-12&2&2\\16&-3&-5\\-8&1&3\end{bmatrix}\begin{bmatrix}9\\52\\0\end{bmatrix}=\frac{-1}{4}\begin{bmatrix}-108+104+0\\144-156-0\\-72+52+0\end{bmatrix}=\frac{1}{-4}\begin{bmatrix}-4\\-12\\-20\end{bmatrix}=\begin{bmatrix}1\\3\\5\end{bmatrix}$$

$$\therefore X=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\3\\5\end{bmatrix}$$

$\therefore$ The solution is x=1, y=3, z=5

7. Solve the equations x+y+z=9, 2x+5y+7z=52, 2x+y-z=0, by Gauss-Jordan method

Sol : The matrix equation corresponding to the given system of equations be AX = D,

The augmented matrix is $[AD] = \begin{bmatrix} 1 & 1 & 1 & 9 \\ 2 & 5 & 7 & 52 \\ 2 & 1 & -1 & 0 \end{bmatrix}$

$$[AD]=\begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 3 & 5 & 34 \\ 0 & -4 & -8 & -52 \end{bmatrix} \begin{matrix} (\because R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_2) \end{matrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 3 & 5 & 34 \\ 0 & 1 & 2 & 13 \end{bmatrix} (\because R_3 \to (-1/4)R_3)$$

$$= \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & 13 \\ 0 & 3 & 5 & 34 \end{bmatrix} (\because R_{32})$$

$$= \begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & -1 & -5 \end{bmatrix} (\because R_3 \to R_3 - 3R_2)$$

$$= \begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & -1 & -5 \end{bmatrix} \begin{matrix} (\because R_1 \to R_1 + R_3) \\ R_2 \to R_2 + 2R_3) \end{matrix}$$

$$[AD] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{bmatrix} \begin{matrix} (\because R_1 \to R_1 - R_2 \\ \\ R_3 \to (-1)R_3 \end{matrix}$$

$\therefore$ from the last augumented matrix, we get x=1, y=3, z=5

8. Solve the equations x+y+2z=1, 2x+y+z=2, x+2y+2z=1 by Gauss-Jordan method

Sol: The augmented matrix is $[AD] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 2 & 2 & 1 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} (R_3 \to R_3 + R_2)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 \to R_1 + R_2 \\ R_2(-1) \end{matrix} \quad \text{.................(I)}$$

Thus [AD] is reduced to Form II with all zeroes in the third row.

Thus Rank (A)=2; Rank[AD]=2 <3 (No. of Unknowns)

$\therefore$ The System is Consistent with Infinite number of solutions.

From (I), we have x=1......(i) , y+z=0.........(ii) $\Rightarrow$ y= –z

Let z=k, k$\in$R then the solution is x=1, y= –k, z=k

9. Solve the equations x+2y–z=3, 3x–y+2z=1, 2x–2y+3z=2, by Gauss-Jordan method

Sol: The matrix equation corresponding to the given system of equations be AX = D,

The augmented matrix is $[AD] = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 6 & 5 & -4 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & -6 & 5 & -4 \end{bmatrix} R_2 \to R_2 - R_3$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 0 & -4 \\ 0 & 0 & 5 & 20 \end{bmatrix} R_3 \to R_3 + 6R_2$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 4 \end{bmatrix} \begin{matrix} R_2 \to R_2(-1) \\ R_3 \to R_3\left(\frac{1}{5}\right) \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 & -5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 4 \end{bmatrix} R_1 \to R_1 - 2R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 4 \end{bmatrix} R_1 \to R_1 + R_3 \qquad \text{..............(I)}$$

Thus [AD] is reduced to Form I of Gauss Jordan Solution.

Hence there exists a Unique solution.

$\therefore$ From (I), the solution is x= –1, y=4, z=4

10. Solve the equations x–3y–8z= –10, 3x+y–4z=0, 2x+5y+6z=13, by Gauss-Jordan method

Sol: The augmented matrix is $[AD] = \begin{bmatrix} 1 & -3 & -8 & -10 \\ 3 & 1 & -4 & 0 \\ 2 & 5 & 6 & 13 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & -3 & -8 & -10 \\ 0 & 10 & 20 & 30 \\ 0 & 11 & 22 & 33 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -3 & -8 & -10 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{matrix} \\ R_2(1/10) \\ R_3(1/11) \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -3 & -8 & -10 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ R_3 \to R_3 - R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 \to R_1 + 4R_2 \\ \\ \\ \end{matrix} \quad \text{............(I)}$$

Thus [AD] is reduced to Form II with all zeroes in the third row.

Thus Rank (A)=2; Rank[AD]=2 <3 (No. of Unknowns)

$\therefore$ The System is Consistent with Infinite number of solutions.

From (I) we have $x + y = 2$ .....(i) , $y + 2z = 3$ ....(ii)

Let $z = k, k \in R$ (ii) $\Rightarrow y = 3 - 2z = 3 - 2k$;

(i) $\Rightarrow x = 2 - y = 2 - (3 - 2k) = 2 - 3 + 2k = 2k - 1$

$\therefore$ The solutions $x = -1 + 2k, y = 3 - 2k, z = k$, $k \in R$