2. DEMOIVRE'S THEOREM

Sections	No. of periods (6)	Weightage in IPE [1x2+1x7=9]
Demoivre's Theorem	6	9 marks

In Mathematics, de Moivre's formula (or) de Moivre's theorem was named after Abraham de Moivre.

This formula connects complex numbers and trigonometry

The generalisation of this formula is used to find explicit expressions for the n^{th} roots of unity.

The applications of De Moivre's theorem are seen in the following

I. Values of $(\cos\theta + i\sin\theta)^{1/n}$

II. Expansions of $\sin n\theta$, $\cos n\theta$, $\tan n\theta$

III. Expression for $\sin^n\theta$, $\cos^n\theta$, $\tan^n\theta$ *interms of multiple angles.*

SYNOPSIS POINTS

1. Demoivre's theorem:

(i) $(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta,\ n\in Z$ (or) $(cis\theta)^n=cis(n\theta)$

(ii) $(\cos\theta+i\sin\theta)^{-n}=\cos n\theta-i\sin n\theta,\ n\in Z$ (or) $(cis\theta)^{-n}=cis(-n\theta)$

(iii) $(\cos\theta+i\sin\theta)^{p/q}=\cos\frac{p}{q}\theta+i\sin\frac{p}{q}\theta,\ \frac{p}{q}\in R,\ q\neq 0$ (or) $(cis\theta)^{p/q}=cis\frac{p}{q}\theta$

2.1. If $x=\cos\theta+i\sin\theta$ then $\frac{1}{x}=\cos\theta-i\sin\theta$ hence (i) $x+\frac{1}{x}=2\cos\theta$ (ii) $x-\frac{1}{x}=2i\sin\theta$

2.2. If $x=\cos\theta+i\sin\theta$ then (i) $x^n+\frac{1}{x^n}=2\cos n\theta$ (ii) $x^n-\frac{1}{x^n}=2i\sin n\theta$

3.1. The roots of $x^3=1$ are called cube roots of unity, which are 1; $\omega=\frac{-1+i\sqrt{3}}{2}$; $\omega^2=\frac{-1-i\sqrt{3}}{2}$

3.2. $1+\omega+\omega^2=0$, hence we have; $1+\omega=-\omega^2$, $1+\omega^2=-\omega$, $\omega+\omega^2=-1$

3.3. $\omega^3=1$, $\omega^4=(\omega^3)\omega=\omega$, $\omega^5=(\omega^3)\omega^2=\omega^2$, $\omega^6=(\omega^3)^2=1$.....

4.1. The roots of $x^4=1$ are called fourth roots of unity, which are $1,-1,i,-i$

4.2. $1=cis0=\cos0+i\sin0$; $-1=cis\pi=\cos\pi+i\sin\pi$;

$i=cis\frac{\pi}{2}=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$; $-i=cis\left(-\frac{\pi}{2}\right)=\cos\frac{\pi}{2}-i\sin\frac{\pi}{2}$

5.1. The n^{th} roots of unity are $cis\frac{2k\pi}{n},\ k=0,1,2,3,...(n-1)$

5.2. The n^{th} roots of $z=r\cos\theta$ are $r^{1/n}cis\left(\frac{2k\pi+\theta}{n}\right),\ k=0,1,2,....(n-1)$

ADDITIONAL QUESTIONS WITH SOLUTIONS

1 **State De Moivre's theorem for integral index.**

Sol: **De Moivre's theorem for integral index:** For a integer 'n' $(\cos\theta+i\sin\theta)^n = \cos n\theta+i\sin n\theta$

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2 **State De Moivre's theorem for rational index.**

Sol: **De Moivre's theorem for rational index:**

For a rational p/q, $(\cos\theta+i\sin\theta)^{p/q} = \cos\frac{p}{q}\theta+i\sin\frac{p}{q}\theta,\ \frac{p}{q}\in Q,\ q\neq 0$

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3 **If 1, ω,ω^2 are the cube roots of unity, then prove that $(x + y + z)(x + y\omega + z\omega^2)(x + y\omega^2 + z\omega) = x^3 + y^3 + z^3-3xyz$**

Sol: Given 1, ω, ω^2 are the cube roots of unity $\Rightarrow 1+\omega+\omega^2 = 0$. Hence $\omega+\omega^2 = -1$.

Also $\omega^3 = 1 \Rightarrow \omega^4 = \omega^3.\omega = 1.\omega=\omega$

Consider, $(x + y\omega + z\omega^2)(x + y\omega^2 + z\omega)$

$$= x^2 + xy\omega^2 + xz\omega + yx\omega + y^2\omega^3 + yz\omega^2 + zx\omega^2 + zy\omega^4 + z^2\omega^3$$

$$= x^2 + y^2(1) + z^2(1) + xy(\omega + \omega^2) + yz(\omega^4 + \omega^2) + zx(\omega + \omega^2)$$

$$= x^2 + y^2 + z^2 + xy(-1) + yz(\omega + \omega^2) + zx(-1)$$

$$= x^2 + y^2 + z^2 - xy - yz - zx$$

Now, L.H.S $= (x + y + z)(x + y\omega + z\omega^2)(x + y\omega^2 + z\omega)$

$$= (x + y + z)(x^2 + y^2 + z^2 - xy-yz - zx)$$
$$= x^3 + y^3 + z^3 - 3xyz = \text{R.H.S}$$

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4 **If $(1+x)^n = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ then show that**

(i) $a_0 - a_2 + a_4.... = 2^{n/2}\cos\frac{n\pi}{4}$ (ii) $a_1 - a_3 + a_5 - .... = 2^{n/2}\sin\frac{n\pi}{4}$

Sol: Given that $(1+x)^n = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ ......(1)

Putting x = i in (1) we get $(1+i)^n = a_0 + a_1i + a_2i^2 + a_3i^3 + ... + a_ni^n$

$$\Rightarrow \left[\sqrt{2}\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)\right]^n = a_0 + a_1.i - a_2 - a_3.i + a_4.1 + a_5.i + ... + a_n.i^n$$

$$\Rightarrow\left[\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]^n=(a_0-a_2+a_4-a_6+a_8....)+i(a_1-a_3+a_5-a_7+....)$$

$$\Rightarrow 2^{n/2}\left(\cos\frac{n\pi}{4}+i\sin\frac{n\pi}{4}\right)=(a_0-a_2+a_4-a_6+a_8....)+i(a_1-a_3+a_5-a_7+....)$$

Equating the real parts, we get $a_0-a_2+a_4-a_6+....=2^{n/2}\cos\frac{n\pi}{4}$

Equating the imaginary parts, we get $a_1-a_3+a_5-a_7+....=2^{n/2}\sin\frac{n\pi}{4}$

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5 **Find the common roots of $x^{12}-1=0$ and $x^4+x^2+1=0$**

Sol: $x^{12}-1=0\Rightarrow(x^6+1)(x^6-1)=(x^6+1)[(x^2)^3-1^3]=(x^6+1)[(x^2-1)(x^4+x^2+1)]=0$

$\therefore$ The common roots of the given two equations are the roots of $x^4+x^2+1=0$

$$x^4+x^2+1=0\Rightarrow x^2=\frac{-1\pm\sqrt{1-4}}{2}=\frac{-1\pm\sqrt{3}i}{2}=\omega \text{ or } \omega^2$$

Now, $x^2=\omega=\omega(1)=\omega(\omega^3)=\omega^4\Rightarrow x=\pm\omega^2$. Also $x^2=\omega^2\Rightarrow x=\pm\omega$

$\therefore$ The common roots are $\pm\omega,\ \pm\omega^2$.

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6 **Find the number of 15^{th} roots of unity, which are also 25^{th} roots of unity.** EAM Q

Sol: The 15^{th} roots of unity are $\text{cis}\frac{2p\pi}{15}$ for p=0,1,2,....14

The 25^{th} roots of unity are $\text{cis}\frac{2q\pi}{25}$ for q=0,1,2,....24

The number of common roots = G.C.D of 15, 25 = 5

Note: For the common roots, we have $\frac{p\pi}{15}=\frac{q\pi}{25}$ ....(1), where p=0,1,2,.....14 and q=0,1,2,....24.

(1) holds true when, p=0 and q=0; p=3 and q=5; p=6 and q=10; p=9 and q=15; p=12 and q=20

Hence, the five common roots are $\left\{\text{cis}0, \text{cis}\frac{2\pi}{5}, \text{cis}\frac{4\pi}{5}, \text{cis}\frac{6\pi}{5}, \text{cis}\frac{8\pi}{5}\right\}$

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7 **If $z^2+z+1=0$, where z is a complex number then prove that**

$$\left(z+\frac{1}{z}\right)^2+\left(z^2+\frac{1}{z^2}\right)^2+\left(z^3+\frac{1}{z^3}\right)^2+\left(z^4+\frac{1}{z^4}\right)^2+\left(z^5+\frac{1}{z^5}\right)^2+\left(z^6+\frac{1}{z^6}\right)^2=12$$

Sol: Comparing $1+z+z^2=0$ with $1+\omega+\omega^2=0$ we get $z=\omega$. (or) solving $z^2+z+1=0$, we get $z=\omega,\omega^2$

If $z=\omega$ then $z+\frac{1}{z}=\omega+\frac{1}{\omega}=\omega+\frac{\omega^3}{\omega}=\omega+\omega^2=-1$; ($\because \omega^3=1$ and $1+\omega+\omega^2=0$)

$$z^2+\frac{1}{z^2}=\omega^2+\frac{1}{\omega^2}=\omega^2+\frac{\omega^3}{\omega^2}=\omega^2+\omega=-1;$$

$$z^3+\frac{1}{z^3}=\omega^3+\omega^3=1+1=2;$$

$$z^4+\frac{1}{z^4}=\omega^4+\frac{1}{\omega^4}=\omega+\frac{1}{\omega}=\omega+\frac{\omega^3}{\omega}=\omega+\omega^2=-1$$

$$z^5+\frac{1}{z^5}=\omega^5+\frac{1}{\omega^5}=\omega^2+\frac{1}{\omega^2}=\omega^2+\frac{\omega^3}{\omega^2}=\omega^2+\omega=-1$$

$$z^6+\frac{1}{z^6}=\omega^6+\frac{1}{\omega^6}=(\omega^3)^2+\frac{1}{(\omega^3)^2}=1+1=2$$

$\therefore$ G.E $=(-1)^2+(-1)^2+2^2+(-1)^2+(-1)^2+2^2=1+1+4+1+1+4=12$

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8 **Prove that the sum of 99th powers of the roots of the equation $x^7-1=0$ is zero and hence deduce the roots of $x^6+x^5+x^4+x^3+x^2+x+1=0$.**

Sol: The given equation is $x^7-1=0$.

The 7 roots of this equation are $1, \alpha, \alpha^2, \ldots \alpha^6$ where $\alpha=\text{cis}\frac{2\pi}{7}$

The above roots are in G.P with common ratio $r=\alpha$ and first term a=1

Now, $1^{99}+\alpha^{99}+(\alpha^2)^{99}+(\alpha^3)^{99}+(\alpha^4)^{99}+(\alpha^5)^{99}+(\alpha^6)^{99}=\frac{1[1-(\alpha^{99})^7]}{1-\alpha^{99}}\left[\because S_n=a\left(\frac{1-r^n}{1-r}\right)\right]$

$$=\frac{1-\left[\text{cis}\frac{2\pi}{7}\right]^{99\times7}}{1-\left[\text{cis}\frac{2\pi}{7}\right]^{99}}=\frac{1-\left[\text{cis}(99\times7)\frac{2\pi}{7}\right]}{1-\left[\text{cis}(99)\frac{2\pi}{7}\right]}=\frac{1-[\text{cis}(99)2\pi]}{1-\left[\text{cis}\frac{198\pi}{7}\right]}$$

$$=\frac{1-(\cos198\pi+i\sin198\pi)}{1-\text{cis}\left(\frac{198\pi}{7}\right)}=\frac{1-(1+i(0))}{1-\text{cis}\left(\frac{198\pi}{7}\right)}=\frac{1-1}{1-\text{cis}\left(\frac{198\pi}{7}\right)}=0$$

Also, $x^7-1=0\Rightarrow(x-1)(x^6+x^5+x^4+x^3+x^2+x+1)=0$

$\therefore$ The roots of $x^6+x^5+x^4+x^3+x^2+x+1=0$ are $\alpha, \alpha^2, \ldots \alpha^6$ where $\alpha=\text{cis}\frac{2\pi}{7}$

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9 **Find the product of all the values of $(1+i)^{4/5}$.**

Sol: $(1+i)^{4/5}=\left[\sqrt{2}\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)\right]^{4/5}=2^{2/5}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^{4/5}=2^{2/5}\left(\text{cis}\frac{\pi}{4}\right)^{4/5}$

$$=2^{2/5}\left(\text{cis}4.\frac{\pi}{4}\right)^{1/5}=2^{2/5}(\text{cis}\pi)^{1/5}$$

$$=2^{2/5}[\text{cis}(2k\pi+\pi)]^{1/5}=2^{2/5}\text{cis}\frac{1}{5}(2k\pi+\pi)$$

$$= 2^{2/5}\,\text{cis}(2k+1)\frac{\pi}{5},\ k = 0,1,2,3,4$$

$\therefore$ Product of all the values $= [2^{2/5}\,\text{cis}\frac{\pi}{5}][2^{2/5}\,\text{cis}\frac{3\pi}{5}][2^{2/5}\,\text{cis}\pi][2^{2/5}\,\text{cis}\frac{7\pi}{5}][2^{2/5}\,\text{cis}\frac{9\pi}{5}]$

$$= (2^{2/5})^5\,\text{cis}\left[\frac{\pi}{5}+\frac{3\pi}{5}+\frac{5\pi}{5}+\frac{7\pi}{5}+\frac{9\pi}{5}\right]$$

$$= 4\text{cis}5\pi = 4(\cos 5\pi + i\sin 5\pi) = 4(-1+i(0)) = -4$$

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10 **Solve $(x-1)^n = x^n$, n is a positive integer.**

Sol: The given equation is $(x-1)^n = x^n$, $x \neq 0$

$$\Rightarrow \frac{(x-1)^n}{x^n} = 1 \Rightarrow \left(\frac{x-1}{x}\right)^n = 1 \Rightarrow \left(\frac{x-1}{x}\right) = 1^{1/n} \Rightarrow \left(1-\frac{1}{x}\right) = 1^{1/n}$$

$$\Rightarrow \frac{1}{x} = 1 - 1^{1/n} = 1 - (\cos 0 + i\sin 0)^{1/n}$$

$$= 1 - [\cos(2k\pi + 0) + i\sin(2k\pi+0)]^{1/n},\ k=0,1,2,.....(n-1)$$

$$= 1 - [\cos(2k\pi) + i\sin(2k\pi)]^{1/n} = 1 - \left(\cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)\right)$$

Applying

$1 - \cos 2A = 2\sin^2 A$

$\sin 2A = 2\sin A\cos A$

$$\Rightarrow \frac{1}{x} = \left[1-\cos\left(\frac{2k\pi}{n}\right)\right] - i\sin\left(\frac{2k\pi}{n}\right) = 2\sin^2\frac{k\pi}{n} - i2\sin\frac{k\pi}{n}\cos\frac{k\pi}{n} = 2\sin\frac{k\pi}{n}\left(\sin\frac{k\pi}{n} - i\cos\frac{k\pi}{n}\right)$$

$$\therefore x = \frac{1}{2\sin\frac{k\pi}{n}\left(\sin\frac{k\pi}{n} - i\cos\frac{k\pi}{n}\right)} = \frac{\sin\frac{k\pi}{n} + i\cos\frac{k\pi}{n}}{2\sin\frac{k\pi}{n}\left(\sin\frac{k\pi}{n} - i\cos\frac{k\pi}{n}\right)\left(\sin\frac{k\pi}{n} + i\cos\frac{k\pi}{n}\right)}$$

$$= \frac{\sin\frac{k\pi}{n} + i\cos\frac{k\pi}{n}}{2\sin\frac{k\pi}{n}\left(\sin^2\frac{k\pi}{n} + \cos^2\frac{k\pi}{n}\right)} = \frac{\sin\frac{k\pi}{n} + i\cos\frac{k\pi}{n}}{2\sin\frac{k\pi}{n}(1)} = \frac{1}{2}\left(\frac{\sin\frac{k\pi}{n}}{\sin\frac{k\pi}{n}} + i\frac{\cos\frac{k\pi}{n}}{\sin\frac{k\pi}{n}}\right) = \frac{1}{2}\left(1 + i\cot\frac{k\pi}{n}\right)$$

where, $k = 0,1,2,........(n-1)$

11 **If $1, \alpha, \alpha^2, \alpha^3, \ldots\ldots\ldots\ldots \alpha^{n-1}$ be the n^{th} roots of unity, then prove that $1^p + \alpha^p + (\alpha^2)^p + (\alpha^3)^p + \ldots + (\alpha^{n-1})^p = 0$; if $p \neq kn$**
$= n$; if $p = kn$, where $p, k \in N$

Sol: **Case(i):** When $p \neq kn$

n^{th} roots of unity are $1, \alpha, \alpha^2, \ldots\ldots\ldots \alpha^{n-1}$ where $\alpha = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$

$$\therefore 1^p + \alpha^p + (\alpha^2)^p + (\alpha^3)^p + \ldots + (\alpha^{n-1})^p$$

$$= 1 + \alpha^p + \alpha^{2p} + \alpha^{3p} + \ldots + (\alpha^{n-1})^p = \frac{1(1-(\alpha^n)^p)}{1-\alpha^p} \left(\because S_n = \frac{a(1-r^n)}{1-r}\right)$$

$$= \frac{1-\alpha^{pn}}{1-\alpha^p} = \frac{1-\left[\cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)\right]^{pn}}{1-\left[\cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)\right]^p}$$

$$= \frac{1-(\cos 2\pi p + i\sin 2\pi p)}{1-\cos\left(\frac{2\pi p}{n}\right) - i\sin\left(\frac{2\pi p}{n}\right)} = \frac{1-1-i(0)}{1-\cos\left(\frac{2\pi p}{n}\right) - i\sin\left(\frac{2\pi p}{n}\right)} = 0 \quad (\because p \neq kn)$$

Case(ii): When $p = kn$

$$\alpha^p = \left[\cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)\right]^p = \cos p\left(\frac{2\pi}{n}\right) + i\sin p\left(\frac{2\pi}{n}\right) = \cos(kn)\left(\frac{2\pi}{n}\right) + i\sin(kn)\left(\frac{2\pi}{n}\right)$$

$$= \cos(2k\pi) + i\sin(2k\pi) = 1 + i(0) = 1$$

Thus each term of the series becomes 1.

$$\therefore 1^p + \alpha^p + (\alpha^2)^p + (\alpha^3)^p + \ldots + (\alpha^{n-1})^p = 1+1+1+\ldots+1 \text{(n times)} = n(1) = n$$