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DIGITAL MATERIAL

PROPERTIES OF TRIANGLES -INDEX

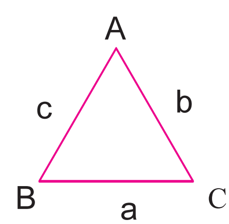
10.PROPERTIES OF TRIANGLES

1. INTRODUCTION PAGE

Sections	No. of periods (15)	Weightage in IPE [1x4+ 1x7 =13]
1. Properties of Triangles-I	9	4 or 7 marks
2. Properties of Triangles-II	6	4 or 7 marks

The three sides and three angles of a triangle are called the six elements of the triangle. When three elements of a triangle are given, atleast one of them being a side, then the remaining three elements can be calculated and this process is known as 'solving a triangle' . For this, we have to depend on some formulae known as sine rule, cosine rule, tangent rule, projection rule etc., Problems related with the sides, angles, half angles, inradius, exradius etc., are dealt with in this topic "Properties of Triangles".

In ΔABC the angles at the vertices A,B,C are denoted by A,B,C respectively. The sides BC, CA, AB are denoted by a,b,c respectively. i.e, a,b,c are the sides opposite to the angles A,B,C respectively.



The perimeter of ΔABC is $2S = a + b + c$, where S is the semi perimeter.

The area of ΔABC is denoted by Δ

The circumradius, inradius of ΔABC are denoted by R, r respectively.

The radii of the excircles opposite to the vertices A,B,C are denoted by r_1, r_2, r_3 respectively.

Some model problems on Heights & Distances are given at the end.

ADDITIONAL QUESTIONS ON PROPERTIES OF TRIANGLES-I

1. Prove that $\sum a^3\sin(B-C)=0$

Sol: $L.H.S=\sum a^2[a\sin(B-C)]=\sum a^2[2R.\sin A\sin(B-C)]$

$$=R\sum a^2\left(2\sin(180^\circ-\overline{B+C})\sin(B-C)\right)=R\sum a^2[2\sin(B+C).\sin(B-C)]$$

$$=2R\sum a^2(\sin^2 B-\sin^2 C)=2R\sum a^2\left(\frac{b^2}{4R^2}-\frac{c^2}{4R^2}\right)$$

$$=\frac{1}{2R}\sum a^2[(b^2-c^2)]=\frac{1}{2R}[a^2(b^2-c^2)+b^2(c^2-a^2)+c^2(a^2-b^2)]$$

$$=\frac{1}{2R}\left(a^2b^2-a^2c^2+b^2c^2-a^2b^2+a^2c^2-b^2c^2\right)=\frac{1}{2R}.0=0=R.H.S$$

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2. Prove that $\frac{a\sin(B-C)}{b^2-c^2}=\frac{b\sin(C-A)}{c^2-a^2}=\frac{c\sin(A-B)}{a^2-b^2}$

Sol: $$\frac{a\sin(B-C)}{b^2-c^2}=\frac{2R.\sin A\sin(B-C)}{4R^2(\sin^2 B-\sin^2 C)}=\frac{\sin(180^\circ-\overline{B+C})\sin(B-C)}{2R\sin(B+C)\sin(B-C)}=\frac{\sin(B+C)}{2R.\sin(B+C)}=\frac{1}{2R}$$

Similary we can show that $\frac{b\sin(C-A)}{c^2-a^2}=\frac{1}{2R};\frac{c\sin(A-B)}{a^2-b^2}=\frac{1}{2R}$

$$\therefore\frac{a\sin(B-C)}{b^2-c^2}=\frac{b\sin(C-A)}{c^2-a^2}=\frac{c\sin(A-B)}{a^2-b^2}$$

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3. Prove that $\sum a^2\frac{\sin(B-C)}{\sin B+\sin C}=0$

Sol: $L.H.S=\sum a^2\left(\frac{\sin(B-C)}{\sin B+\sin C}\right)=\sum a\left(\frac{a\sin(B-C)}{\sin B+\sin C}\right)$

$$=\sum a\frac{2R.\sin A.\sin(B-C)}{\sin B+\sin C}=\sum a\frac{2R.\sin(B+C)\sin(B-C)}{\sin B+\sin C}=\sum a\frac{2R.(\sin^2 B-\sin^2 C)}{\sin B+\sin C}$$

$$=\sum a\frac{2R.(\sin B-\sin C)(\sin B+\sin C)}{(\sin B+\sin C)}=\sum a.2R(\sin B-\sin C)=\sum a.(2R\sin B-2R\sin C)$$

$$=\sum a(b-c)=a(b-c)+b(c-a)+c(a-b)=ab-ac+bc-ab+ca-bc=0$$

4. Prove that $(b-a)\cos C + c(\cos B - \cos A) = c.\sin\left(\frac{A-B}{2}\right)\operatorname{cosec}\left(\frac{A+B}{2}\right)$

Sol: L.H.S. = bcosC – acosC + ccosB –c cosA

= (bcosC + ccosB) –(acosC + ccosA)= a–b=2R(sinA–sinB)

$$= 2R\left(2\cos\frac{A+B}{2}\sin\frac{A-B}{2}\right) = 4R\sin\frac{A-B}{2}\cos\left(90^o - \frac{C}{2}\right)$$

$$= 4R\sin\left(\frac{A-B}{2}\right)\sin\frac{C}{2} = \frac{2R\left(2\sin\frac{C}{2}\cos\frac{C}{2}\right)\sin\left(\frac{A-B}{2}\right)}{\cos\frac{C}{2}}$$

$$= \frac{2R\sin C\sin\left(\frac{A-B}{2}\right)}{\cos\left(90^o - \frac{A+B}{2}\right)} = \frac{c\sin\left(\frac{A-B}{2}\right)}{\sin\left(\frac{A+B}{2}\right)} = c\sin\left(\frac{A-B}{2}\right)\operatorname{cosec}\left(\frac{A+B}{2}\right) = \text{R.H.S}$$

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5. Show that (i) $\Sigma(a+b)\tan\left(\frac{A-B}{2}\right) = 0$ **(ii)** $\frac{b-c}{b+c}\cot\frac{A}{2} + \frac{b+c}{b-c}\tan\frac{A}{2} = 2\operatorname{cosec}(B-C)$

Sol: **(i)** $\sum[2R\sin A + 2R\sin B]\tan\left(\frac{A-B}{2}\right)$

$$= \sum 2R(\sin A + \sin B).\tan\left(\frac{A-B}{2}\right)$$

$$= \sum 2R\left[2\sin\left(\frac{A+B}{2}\right).\cos\left(\frac{A-B}{2}\right)\right].\frac{\sin\left(\frac{A-B}{2}\right)}{\cos\left(\frac{A-B}{2}\right)}$$

$$= \sum 4R\left[\sin\left(\frac{A}{2}+\frac{B}{2}\right)\sin\left(\frac{A}{2}-\frac{B}{2}\right)\right] = \sum 4R\left[\sin^2\frac{A}{2} - \sin^2\frac{B}{2}\right]$$

$$= 4R\left[\left(\sin^2\frac{A}{2} - \sin^2\frac{B}{2}\right) + \left(\sin^2\frac{B}{2} - \sin^2\frac{C}{2}\right) + \left(\sin^2\frac{C}{2} - \sin^2\frac{A}{2}\right)\right] = 4R(0) = 0 = \text{R.H.S.}$$

(ii) From Napier's formula

$$\frac{b-c}{b+c}\cot\frac{A}{2} = \tan\left(\frac{B-C}{2}\right) \Rightarrow \frac{b+c}{b-c}\tan\frac{A}{2} = \frac{1}{\tan\left(\frac{B-C}{2}\right)}$$

$$\text{L.H.S} = \frac{b-c}{b+c}\cot\frac{A}{2} + \frac{b+c}{b-c}\tan\frac{A}{2} = \tan\left(\frac{B-C}{2}\right) + \frac{1}{\tan\left(\frac{B-C}{2}\right)}$$

$$= \frac{1+\tan^2\left(\frac{B-C}{2}\right)}{\tan\left(\frac{B-C}{2}\right)} = \frac{2}{\frac{2\tan\left(\frac{B-C}{2}\right)}{1+\tan^2\left(\frac{B-C}{2}\right)}} = \frac{2}{\sin(B-C)} = 2\,\text{cosec}(B-C) = \text{R.H.S}$$

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6. If C=90° then prove that $\frac{a^2+b^2}{a^2-b^2}\sin(A-B)=1$ **EAM Q**

Sol: Given that C=90° then $c^2=a^2+b^2$

$$\text{L.H.S.} = \frac{a^2+b^2}{a^2-b^2}\sin(A-B) = \frac{c^2}{a^2-b^2}\sin(A-B) = \frac{4R^2\sin^2 C}{4R^2(\sin^2 A-\sin^2 B)}\sin(A-B)$$

$$= \frac{\sin^2 C}{\sin^2 A-\sin^2 B}\sin(A-B) = \frac{\sin^2 C}{\sin(A+B)\,\cancel{\sin(A-B)}}\cancel{\sin(A-B)}$$

$$= \frac{\sin^2 C}{\sin(A+B)} = \frac{\sin C.\sin C}{\sin(A+B)} = \frac{\sin 90^\circ\ \cancel{\sin(A+B)}}{\cancel{\sin(A+B)}} = \sin 90^\circ = 1$$

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7. Prove that $(b+c)\cos\left(\frac{B+C}{2}\right) = a\cos\left(\frac{B-C}{2}\right)$

Sol: Consider $\frac{b+c}{a} = \frac{\cancel{2R}\sin B+\cancel{2R}\sin C}{\cancel{2R}\sin A} = \frac{\sin B+\sin C}{\sin A} = \frac{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}{2\sin\frac{A}{2}\cos\frac{A}{2}}$

$$= \frac{\cancel{\sin\left(\frac{B+C}{2}\right)}\cos\left(\frac{B-C}{2}\right)}{\cos\left(\frac{B+C}{2}\right)\cancel{\sin\left(\frac{B+C}{2}\right)}} = \frac{\cos\left(\frac{B-C}{2}\right)}{\cos\left(\frac{B+C}{2}\right)}$$

$$\therefore \frac{b+c}{a} = \frac{\cos\left(\frac{B-C}{2}\right)}{\cos\left(\frac{B+C}{2}\right)} \Rightarrow (b+c)\cos\left(\frac{B+C}{2}\right) = a\cos\left(\frac{B-C}{2}\right)$$

ADDITIONAL QUESTIONS ON PROPERTIES OF TRIANGLES-II

1. Prove that $\Sigma(r+r_1)\tan\left(\frac{B-C}{2}\right)=0$

Sol: $r+r_1=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}+4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

$=4R\sin\frac{A}{2}\left[\sin\frac{B}{2}\sin\frac{C}{2}+\cos\frac{B}{2}\cos\frac{C}{2}\right] \quad =4R\sin\frac{A}{2}\cos\left(\frac{B-C}{2}\right)$

$\Rightarrow (r+r_1)\tan\left(\frac{B-C}{2}\right)=4R\sin\frac{A}{2}\cos\left(\frac{B-C}{2}\right)\left[\frac{\sin\left(\frac{B-C}{2}\right)}{\cos\left(\frac{B-C}{2}\right)}\right]$

$=4R\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right) \quad =2R(\sin B-\sin C)=b-c$

Hence $\Sigma(r+r_1)\tan\left(\frac{B-C}{2}\right)=\Sigma(b-c)=b-c+c-a+a-b=0$

EAM Q

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2. Show that $(r_1+r_2)\tan\frac{C}{2}=(r_3-r)\cot\frac{C}{2}=c.$

Sol: Consider $(r_1+r_2)\tan\frac{C}{2}=\left[4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}+4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}\right]\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}$

$=4R\left[\sin\frac{A}{2}\cos\frac{B}{2}+\cos\frac{A}{2}\sin\frac{B}{2}\right]\sin\frac{C}{2} \quad =4R\sin\left(\frac{A+B}{2}\right)\sin\frac{C}{2} \quad =4R\sin\left(90^o-\frac{C}{2}\right)\sin\frac{C}{2}$

$=4R\cos\frac{C}{2}\sin\frac{C}{2} \quad =2R\left[2\sin\frac{C}{2}\cos\frac{C}{2}\right] \quad =2R\sin C=c$ ......(1)

Now consider $(r_3-r)\cot\frac{C}{2}=\left[4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}-4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right]\frac{\cos\frac{C}{2}}{\sin\frac{C}{2}}$

$=4R\left[\cos\frac{A}{2}\cos\frac{B}{2}-\sin\frac{A}{2}\sin\frac{B}{2}\right]\cos\frac{C}{2} \quad =4R\cos\left(\frac{A+B}{2}\right)\cos\frac{C}{2}$

$=4R\cos\left(90^o-\frac{C}{2}\right)\cos\frac{C}{2} \quad =4R\sin\frac{C}{2}\cos\frac{C}{2} \quad =2R\left[2\sin\frac{C}{2}\cos\frac{C}{2}\right] \quad =2R\sin C=c$ ......(2)

$\therefore$ From(1) & (2), $(r_1+r_2)\tan\frac{C}{2}=(r_3-r)\cot\frac{C}{2}=c.$

3. **Show that i)** $a=(r_2+r_3)\sqrt{\frac{rr_1}{r_2r_3}}$ **ii)** $\Delta=r_1r_2\sqrt{\frac{4R-r_1-r_2}{r_1+r_2}}$

Sol: (i) Consider $\frac{rr_1}{r_2.r_3}=\frac{\frac{\Delta}{s}.\frac{\Delta}{s-a}}{\frac{\Delta}{s-b}.\frac{\Delta}{s-c}}=\frac{(s-b)(s-c)}{s(s-a)}$

Now $r_2+r_3=\frac{\Delta}{s-b}+\frac{\Delta}{s-c}=\frac{\Delta[(s-b)+(s-c)]}{(s-b)(s-c)}=\frac{\Delta[(2s-b-c)]}{(s-b)(s-c)}=\frac{\Delta(a+b+c-b-c)}{(s-b)(s-c)}=\frac{\Delta.a}{(s-b)(s-c)}$

$$R.H,.S=\frac{\Delta a}{(s-b)(s-c)}\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=\frac{\Delta a}{\sqrt{s(s-a)(s-b)(s-c)}}=\frac{\Delta a}{\Delta}=a=L.H.S$$

(ii) $R.H.S=r_1r_2\sqrt{\frac{4R-r_1-r_2}{r_1+r_2}}=r_1r_2\sqrt{\frac{4R-(r_1+r_2)}{r_1+r_2}}=r_1r_2\sqrt{\frac{4R-4R\cos^2\frac{C}{2}}{4R\cos^2\frac{C}{2}}}\left[\because r_1+r_2=4R\cos^2\frac{c}{2}\right]$

$$=r_1r_2\sqrt{\frac{4R\left(1-\cos^2\frac{C}{2}\right)}{4R\cos^2\frac{C}{2}}}=r_1r_2\sqrt{\frac{\sin^2\frac{C}{2}}{\cos^2\frac{C}{2}}}=\frac{\Delta}{s-a}\frac{\Delta}{s-b}\tan\frac{C}{2}$$

$$=\frac{\Delta^2}{(s-a)(s-b)}\sqrt{\frac{(s-b)(s-a)}{s(s-c)}}=\frac{\Delta^2}{\sqrt{s(s-a)(s-b)(s-c)}}=\frac{\Delta^2}{\Delta}=\Delta=L.H.S$$

ADDITIONAL QUESTIONS ON PROPERTIES OF TRIANGLES-III

HEIGHTS & DISTANCES

1. Two trees A and B are on the same side of a river. From a point C in the river the distances of the trees A and B are 250m and 300m respectively. If the angle C is 45º, find the distance between the trees (use $\sqrt{2} = 1.414$)

Sol: Note that triangle ABC may not be a right angled triangle.

Hence, applying cosine rule on ΔABC, we have

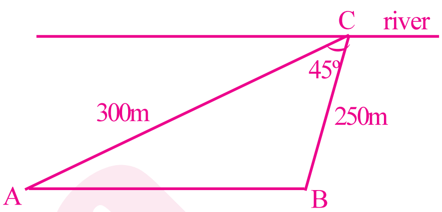
$AB^2 = BC^2+AC^2-2BC.AC\cos45°$

$\Rightarrow AB^2 = 250^2 + 300^2 - 2(250)(300)\cos45°$

$= (25\times10)^2+(30\times10)^2+2(25)(10)(30)(10).\dfrac{1}{\sqrt{2}}$

$= 100(625+900-750\sqrt{2}) \cong 46450.$

$\therefore AB \cong \sqrt{46450} = 215.5\text{ m}$



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2. The upper 3/4th portion of a vertical pole subtends an angle $\text{Tan}^{-1}(3/5)$ at a point in the horizontal plane through its foot and at a distance 40m from the foot. Given that the vertical pole is at a height less than 100m from the ground, find its height.

Sol: Let AB denote the pole of height h; BD=3h/4, AD=h/4. Also, AC=40

Given $\angle BCD = \text{Tan}^{-1}\dfrac{3}{5} = \theta \Rightarrow \tan\theta = \dfrac{3}{5}$

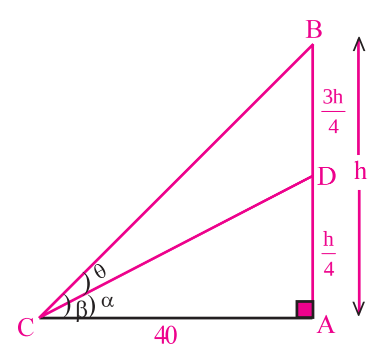
Let $\angle DCA = \alpha$ and $\angle BCA = \beta$

$\therefore \beta=\theta+\alpha \Rightarrow \theta=\beta-\alpha$

From ΔDCA, $\tan\alpha = \dfrac{AD}{AC} = \dfrac{\frac{h}{4}}{40} = \dfrac{h}{160}$

From ΔBCA, $\tan\beta = \dfrac{AB}{AC} = \dfrac{h}{40}$

$\therefore \tan\theta=\tan(\beta-\alpha) = \dfrac{\tan\beta - \tan\alpha}{1-\tan\beta\tan\alpha}$



$$\Rightarrow \frac{3}{5} = \frac{\frac{h}{40}-\frac{h}{160}}{\left(\frac{h}{40}\right)\left(\frac{h}{160}\right)} = \frac{\frac{4h-h}{160}}{\frac{6400+h^2}{6400}} \Rightarrow \frac{3}{5} = \frac{3h}{160}\times\frac{6400}{6400+h^2} \Rightarrow \frac{1}{5} = \frac{40h}{6400+h^2}$$

$\Rightarrow 6400 + h^2 = 200h \Rightarrow h^2-200h+6400=0$

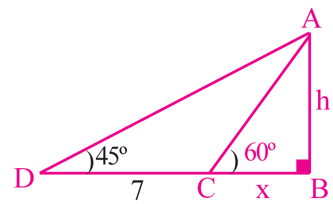
$\Rightarrow h^2-160h-40h+6400=0 \Rightarrow h(h-160)-40(h-160)=0$

$\Rightarrow (h-160)(h-40)=0 \Rightarrow h = 40$ or 160

But given that, height of pole is less than 100 m $\therefore h=40m$

3. AB is a vertical pole with B at the ground level and and A at the top. A man finds that the angle of elevation of the point A from a certain point C on the ground is 60°. He moves away from the pole along the line BC to a point D such that CD=7m. From D, the angle of elevation of the point A is 45°. Find the height of the pole.

Sol: Let AB denote the pole of height 'h'



Given CD=7

$\angle ACB = 60^\circ, \angle ADB = 45^\circ$

Let BC=x

From ΔACB, $\tan 60^\circ = h/x$

$$\Rightarrow \sqrt{3} = \frac{h}{x} \Rightarrow x = \frac{h}{\sqrt{3}} \quad ....(1)$$

$$\text{From } \Delta ADB,\ \tan 45^\circ = \frac{h}{7+x} \Rightarrow 1 = \frac{h}{7+x} \Rightarrow h = x+7$$

$$(1) \Rightarrow h = \frac{h}{\sqrt{3}} + 7 \Rightarrow h\left(1 - \frac{1}{\sqrt{3}}\right) = 7$$

$$\Rightarrow h\left(\frac{\sqrt{3}-1}{\sqrt{3}}\right) = 7 \Rightarrow h = \frac{7\sqrt{3}}{\sqrt{3}-1} = \frac{7\sqrt{3}}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1} = \frac{7\sqrt{3}(\sqrt{3}+1)}{2} \text{ m}$$

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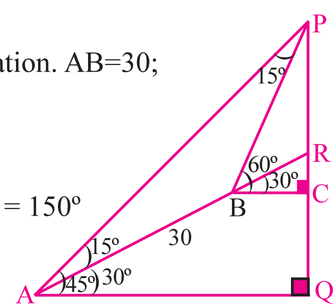
4. The angle of elevation of the top point P of the vertical tower PQ of height h from a point A is 45° and from a point B is 60°, where B is a point at a distance 30 meters from the point A measured along the line AB which makes an angle 30° with AQ. Find the height of the tower.



Sol: PQ is the tower of height h. A and B are the points of observation. AB=30;

$\angle PAQ = 45^\circ$; $\angle BAQ = 30^\circ$. Hence $\angle PAB = 45^\circ - 30^\circ = 15^\circ$

Let the line extension of AB meets PQ at R.

From ΔABP, $\angle PBR = 60^\circ - 30^\circ = 30^\circ$ $\therefore \angle ABP = 180^\circ - 30^\circ = 150^\circ$

$\therefore \angle APB = 180^\circ - (\angle PAB + \angle ABP) = 180^\circ - (15^\circ + 150^\circ) = 15^\circ$

Applying sine rule on ΔPAB;

$$\frac{30}{\sin 15^\circ} = \frac{AP}{\sin 150^\circ} \Rightarrow AP = \frac{30 \times \sin 150^\circ}{\sin 15^\circ} = 30 \times \frac{1}{2} \times \frac{2\sqrt{2}}{\sqrt{3}-1} = \frac{30\sqrt{2}}{\sqrt{3}-1}$$

$$\text{From } \Delta PAQ;\ \sin 45^\circ = \frac{PQ}{PA} \Rightarrow \frac{1}{\sqrt{2}} = \frac{PQ}{PA} \Rightarrow PQ = \frac{PA}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} \frac{30\sqrt{2}}{\sqrt{3}-1} = \frac{30}{\sqrt{3}-1} = \frac{30(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)} = \frac{30(\sqrt{3}+1)}{2} = 15(\sqrt{3}+1)\text{m}$$