WELCOME

STAR 'QR CODE'

DIGITAL MATERIAL

APPLICATIONS OF DERIVATIVES - INDEX

10. APPLICATIONS OF DERIVATIVES

ADDITIONAL QUESTIONS WITH SOLUTIONS

1. Find the points at which the curve y = sin x has horizontal tangents.

Sol: Given $y = \sin x \Rightarrow \frac{dy}{dx} = \cos x \Rightarrow$ Slope at any point on the curve $= \cos x$

A tangent is horizontal if its slope is $0 \Rightarrow \cos x = 0 \Rightarrow x = (2n+1)\frac{\pi}{2}, n \in Z$

When $x = (2n+1)\frac{\pi}{2}, n \in Z$ we have $y = \sin(2n+1)\frac{\pi}{2} = (-1)^n, n \in Z$.

$\therefore$ The required points on the curve $= \left((2n+1)\frac{\pi}{2}, (-1)^n\right), n \in Z$.

---

2. If the slope of the tangent to the curve $x^2-2xy+4y=0$ at a point on it is $-3/2$, then find the equations of tangent and normal at that point.

Sol: Given equation is $x^2-2xy+4y=0$..........(1) Diff. (1) w.r.t x, we have

$$2x - 2x\frac{dy}{dx} - 2y + 4\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx}(4-2x) = 2y - 2x \Rightarrow \frac{dy}{dx} = \frac{2(y-x)}{2(2-x)} = \frac{x-y}{x-2}$$

Given slope $= \frac{-3}{2}$

$$\Rightarrow \frac{x-y}{x-2} = \frac{-3}{2} \Rightarrow 2x - 2y = -3x + 6 \Rightarrow 5x - 2y = 6 \Rightarrow 2y = 5x - 6 \text{ ....(2)}$$

Solving (1) &(2) we get the points of intersection

From (1), $x^2 - x(5x-6) + 2(5x-6) = 0 \Rightarrow x^2 - 5x^2 + 6x + 10x - 12 = 0 \Rightarrow -4x^2 + 16x - 12 = 0$

$\Rightarrow -4(x^2 + 4x + 3) = 0 \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow (x-1)(x-3) = 0 \Rightarrow x = 1$ or $x=3$

If x=1 then (1) $\Rightarrow x^2-2xy+4y=0 \Rightarrow 1-2y+4y=0 \Rightarrow 2y=-1 \Rightarrow y = -\frac{1}{2}$

$\therefore$ Point of contact is $P\left(1, \frac{-1}{2}\right)$

If x=3 then (1) $\Rightarrow x^2-2xy+4y=0 \Rightarrow 9-6y+4y=0 \Rightarrow 2y=9 \Rightarrow y = \frac{9}{2}$

$\therefore$ Another Point of contact is $Q\left(3, \frac{9}{2}\right)$

Case (i): At $P\left(1,-\frac{1}{2}\right)$

Given slope of the tangent is $-\frac{3}{2}$. Hence the slope of the normal is $\frac{2}{3}$

Now, equation of the tangent at $P\left(1,-\frac{1}{2}\right)$ with slope $-\frac{3}{2}$ is $y+\frac{1}{2}=\frac{-3}{2}(x-1)$

$\Rightarrow \frac{2y+1}{2}=\frac{-3(x-1)}{2} \Rightarrow 2y-1=-3x+3 \Rightarrow 3x+2y-2=0$

Also equation of the normal at $P\left(1,-\frac{1}{2}\right)$ with slope $\frac{2}{3}$ is $y+\frac{1}{2}=\frac{2}{3}(x-1)$

$\Rightarrow \frac{2y+1}{2}=\frac{2}{3}(x-1) \Rightarrow 3(2y+1)=4(x-1) \Rightarrow 6y+3=4x-4 \Rightarrow 4x-6y-7=0$

Case (ii): At $Q\left(3,\frac{9}{2}\right)$

Equation of the tangent at $Q\left(3,\frac{9}{2}\right)$ with slope $-\frac{3}{2}$ is $y-\frac{9}{2}=\frac{-3}{2}(x-3)$

$\Rightarrow \frac{2y-9}{2}=\frac{-3}{2}(x-3) \Rightarrow 2y-9=-3x+9 \Rightarrow 3x+2y-18=0$

Also equation of the normal at $Q\left(3,\frac{9}{2}\right)$ with slope $\frac{2}{3}$ is $y-\frac{9}{2}=\frac{2}{3}(x-3)$

$\Rightarrow 3(2y-9)=4(x-3) \Rightarrow 6y-27=4x-12 \Rightarrow 4x-6y+15=0$

---

3. If the slope of the tangent to the curve y=xlogx at a point on it is 3/2, then find the equations of tangent and normal at that point.

Sol: Given equation is y=xlogx ....(1) Diff. w.r.t x using uv formula, we have

$$\frac{dy}{dx}=x\left(\frac{1}{x}\right)+\log x(1)=1+\log x$$

Given slope $=\frac{3}{2}$

$$\therefore 1+\log x=\frac{3}{2} \Rightarrow \log x=\frac{3}{2}-1 \Rightarrow \log x=\frac{3-2}{2}=\frac{1}{2}$$

$$\therefore \log_e x=\frac{1}{2} \Rightarrow x=e^{\frac{1}{2}}=\sqrt{e}$$

From (1), $y=\sqrt{e}\log_e\sqrt{e}=\sqrt{e}\log_e e^{\frac{1}{2}}=\sqrt{e}\frac{1}{2}(\log_e e)=\frac{\sqrt{e}}{2}(1)=\frac{\sqrt{e}}{2}$

$\therefore$ Point of contact is (x,y)= $\left(\sqrt{e},\frac{\sqrt{e}}{2}\right)$

(i) Equation of the tangent at $\left(\sqrt{e},\frac{\sqrt{e}}{2}\right)$ with slope $\frac{3}{2}$ is $y-\frac{\sqrt{e}}{2}=\frac{3}{2}(x-\sqrt{e})$

$\Rightarrow \frac{2y-\sqrt{e}}{2}=\frac{3}{2}(x-\sqrt{e}) \Rightarrow 2y-\sqrt{e}=3x-3\sqrt{e} \Rightarrow 3x-2y-2\sqrt{e}=0$

(ii) Also slope of the tangent is $\frac{3}{2} \Rightarrow$ Slope of the normal is $-\frac{2}{3}$

Equation of the Normal at $\left(\sqrt{e},\frac{\sqrt{e}}{2}\right)$ with slope $-\frac{2}{3}$ is $y-\frac{\sqrt{e}}{2}=-\frac{2}{3}(x-\sqrt{e})$

$\Rightarrow 3(2y-\sqrt{e})=-4(x-\sqrt{e}) \Rightarrow 6y-3\sqrt{e}=-4x+4\sqrt{e} \Rightarrow 4x+6y-7\sqrt{e}=0$.

---

4. Find whether the curve $y = f(x) = x^{2/3}$ has a vertical tangent at x = 0.

Sol: Given that $y = f(x) = x^{2/3}$

$$\text{L.H.D} = \underset{h\to 0-}{\text{Lt}} \frac{f(0+h)-f(0)}{h} = \underset{h\to 0-}{\text{Lt}} \frac{h^{2/3}}{h} = \underset{h\to 0-}{\text{Lt}} \frac{1}{h^{1/3}} = -\infty$$

$$\text{R.H.D} = \underset{h\to 0+}{\text{Lt}} \frac{f(0+h)-f(0)}{h} = \underset{h\to 0+}{\text{Lt}} \frac{h^{2/3}}{h} = \underset{h\to 0+}{\text{Lt}} \frac{1}{h^{1/3}} = \infty$$

Here L.H.D $\neq$ R.H.D

$\therefore$ Vertical tangent does not exist at the point x = 0.

$\sqrt[3]{-ve} = -ve$
$\sqrt[3]{+ve} = +ve$

---

5. Verify whether the curve $y = f(x) = x^{1/3}$ has a vertical tangent at the point with x = 0.

Sol: Given that $y = f(x) = x^{1/3}$

$$f'(0) = \underset{h\to 0}{\text{Lt}} \frac{f(0+h)-f(0)}{h} = \underset{h\to 0}{\text{Lt}} \frac{h^{1/3}}{h} = \underset{h\to 0}{\text{Lt}} \frac{I}{h^{2/3}} = \underset{h\to 0}{\text{Lt}} \frac{1}{(h^{1/3})^2} = \infty$$

$\sqrt[3]{(-ve)^2} = +ve$
$\sqrt[3]{(+ve)^2} = +ve$

$\therefore$ The function has a vertical tangent at the point whose x co-ordinate is 0.

6. A manufacturer can sell x items at a price of rupees $\left(5-\frac{x}{100}\right)$ each. The cost of price of x items is Rs. $\left(\frac{x}{5}+500\right)$. Find the number of items that the manufacturer should sell to earn maximum profits.

Sol: Let selling price of x items is $S(x)$ and the cost price of x items is $C(x)$.

Then, we have S(x)= (cost of each item) x. Also $C(x)=\left(\frac{x}{5}+500\right)$

Now $S(x)=\left(5-\frac{x}{100}\right)x=5x-\frac{x^2}{100}$

Let P (x) denote the profit function. Then

$$P(x)=S(x)-C(x)=\left(5x-\frac{x^2}{100}\right)-\left(\frac{x}{5}+500\right)=\left(\frac{24x}{5}\right)-\left(\frac{x^2}{100}\right)-500............(1)$$

For maxima or minima $\frac{dP(x)}{dx}=0\Rightarrow\frac{24}{5}-\frac{x}{50}=0\Rightarrow\frac{x}{50}=\frac{24}{5}\Rightarrow x=\frac{24\times 50}{5}=240$

The stationary point of P(x) is x= 240. Also $\left[\frac{d^2P(x)}{dx^2}\right]=-\frac{1}{50}\,\forall x$

$\therefore$ The number of items that the manufacturer should sell to earn maximum profit is 240.

---

7. Find the absolute maximum value of $x^{40}-x^{20}$ on the interval [0, 1]. Also, find its absolute minimum value on this interval.

Sol: Let $f(x)=x^{40}-x^{20}$

Then $f'(x)=40x^{39}-20x^{19}=20x^{19}(2x^{20}-1)$.

If $f'(x)=0$ then $20x^{19}(2x^{20}-1)=0\ \Rightarrow x=0$ or $x=\left(\frac{1}{2}\right)^{\frac{1}{20}}$.

Now consider the end points 0 and 1 of the given interval [0,1]

Altogether we get three points 0, 1, $(1/2)^{1/20}$

Now, f(0)=0 and f(1)=0

Also, $f\left(\left(\frac{1}{2}\right)^{\frac{1}{20}}\right)=\left(\frac{1}{2}\right)^{\frac{40}{20}}-\left(\frac{1}{2}\right)^{\frac{20}{20}}=\left(\frac{1}{2}\right)^2-\frac{1}{2}=\frac{1}{4}-\frac{1}{2}=\frac{-1}{4}$

The maximum value of f over [0,1]=max. $\left\{0,\frac{-1}{4},0\right\}=0$, which is attained at x=0 or x=1

$\therefore$ Absolute maximum value is 0

The minimum value of f over $[0,1]=\min.\left\{0,\frac{-1}{4},0\right\}=\frac{-1}{4}$ which is attained at $x=\left(\frac{1}{2}\right)^{\frac{1}{20}}$

$\therefore$ Absolute minimum value is $-1/4$

8. Show that $\frac{x}{1+x} < \ln(1+x) < x$ when $x > 0$

Sol: Let $f(x) = \log(1+x) - \frac{x}{1+x}$ ....(1)

$\Rightarrow f'(x) = \frac{1}{1+x} - \left(\frac{1-0}{(x+1)^2}\right) = \frac{1+x-1}{(1+x)^2} = \frac{x}{(1+x)^2} > 0$ for $x>0$

$\Rightarrow$ f(x) is increasing for x>0

Also, (1) $\Rightarrow$ f(0)=log(1+0)–0=log1–0=0–0=0 $\Rightarrow$ f(0)=0

Now, f(x) is increasing for x>0 $\Rightarrow$f(x)>f(0) $\Rightarrow$ f(x)>0

$\therefore (1) \Rightarrow \log(1+x) - \frac{x}{1+x} > 0 \Rightarrow \log(1+x) > \frac{x}{1+x}$.............(2)

Let g(x)=x–log(1+x) ... (3)

$\Rightarrow g'(x) = 1 - \frac{1}{1+x} = \frac{1+x-1}{1+x} = \frac{x}{1+x} > 0,$ for $x > 0 \Rightarrow$ g(x) is increasing for x>0

Now, (3) $\Rightarrow$ g(0)=0–log(1+0)=0–log1=0–0=0 i.e, g(0)=0

Now, g(x) is increasing for x>0 $\Rightarrow$ g(x) > g(0) $\Rightarrow$ g(x)>0

(3) $\Rightarrow$ x–log(1+x)>0 $\Rightarrow$ x>log(1+x) ..... (4)

$\therefore$ from (2), (4) , $x > \log(1+x) > \frac{x}{1+x} \Rightarrow \frac{x}{1+x} < \log(1+x) < x$

---

9. Find the absolute extremum of $f(x) = 4x - \frac{x^2}{2}$ on $\left[-2, \frac{9}{2}\right]$.

Sol: Given $f(x) = 4x - \frac{x^2}{2} \Rightarrow f'(x) = 4 - x$

For maxima or minima $f'(x) = 0 \Rightarrow 4 - x = 0 \Rightarrow x = 4$

Now the critical point is 4 and end points of $\left[-2, \frac{9}{2}\right]$ are $-2, \frac{9}{2}$

(i) $f(4) = 16 - \frac{16}{2} = 8$

(ii) $f(-2) = -8 - \frac{4}{2} = -8 - 2 = -10$

(iii) $f\left(\frac{9}{2}\right) = 4\left(\frac{9}{2}\right) - \frac{1}{2}\left(\frac{9}{2}\right)^2 = 18 - \frac{81}{4} = \frac{72-81}{4} = \frac{-9}{4}$

$\therefore$ From (1), (2) & (3) Absolute minimum = –10 and Absolute maximum = 8

10. Use the second derivative test to find local extrema of $f(x) = -x^3 + 12x^5 - 5$ on R.

Sol: Given $f(x) = -x^3 + 12x^2 - 5 \Rightarrow f'(x) = -3x^2 + 24x = -3x(x-8) \Rightarrow f''(x) = -6x + 24$

At maxima or minima $f'(x)=0 \Rightarrow x = 0, 8$

At $x = 0$, $f''(0) = 24 > 0 \Rightarrow f(x)$ has minimum value at $x=0$

Minimum value is $f(0) = -5$

At $x = 8$, $f''(8) = -24 < 0 \Rightarrow f(x)$ has maximum value at $x=8$

Maximum value is $f(8) = -8^3 + 12(8)^2 - 5 = 251$

Local minimum $= -5$ & Local maximum $= 251$

---

11. At what points the slopes of the tangents to $y = \frac{x^3}{6} - \frac{3x^2}{2} + \frac{11x}{2} + 12$ increases?

Sol: Equation of the curve is $y = \frac{x^3}{6} - \frac{3}{2}x^2 + \frac{11x}{2} + 12$

$\Rightarrow$Slope m= $\frac{dy}{dx} = \frac{3x^2}{6} - \frac{3}{2}(2x) + \frac{11}{2} = \frac{x^2}{2} - 3x + \frac{11}{2}$

$\therefore$ Slope $m = \frac{x^2}{2} - 3x + \frac{11}{2}$.

Diff. (m) w.r.t x we have $\frac{dm}{dx} = \frac{2x}{2} - 3 = x - 3$

Slope m increases when $\frac{dm}{dx} > 0 \Rightarrow x - 3 > 0 \Rightarrow x > 3$

$\therefore$ Slopes of tangents increase in $(3, \infty)$.

---

12. Determine the intervals in which $f(x) = \frac{2}{(x-1)} + 18x \ \forall \ x \in R \setminus \{0\}$ is strictly increasing and decreasing.

Sol: Let $f(x) = \frac{2}{x-1} + 18x \Rightarrow f'(x) = -\frac{2}{(x-1)^2} + 18 = \frac{-2 + 18(x-1)^2}{(x-1)^2} = \frac{18x^2 - 36x + 16}{(x-1)^2}$

$$f'(x) > 0 \Rightarrow \frac{18x^2 - 36x + 16}{(x-1)^2} > 0 \Rightarrow \frac{2(3x-2)(3x-4)}{(x-1)^2} > 0 \Rightarrow (3x-2)(3x-4) > 0 \Rightarrow x \in \left(-\infty, \frac{2}{3}\right) \cup \left(\frac{4}{3}, \infty\right)$$

$$f'(x) < 0 \Rightarrow \frac{18x^2 - 36x + 16}{(x-1)^2} < 0 \Rightarrow \frac{2(3x-2)(3x-4)}{(x-1)^2} < 0 \Rightarrow (3x-2)(3x-4) < 0 \Rightarrow x \in \left(\frac{2}{3}, \frac{4}{3}\right)$$

$\therefore$ f(x) is increasing on $(-\infty, 2/3) \cup (4/3, \infty)$ and decreasing on $(2/3, 4/3)$.

13. Let f(x)=(x–1)(x–2)(x–3) then prove that there is more than one 'c' in (1,3) such that f '(c)=0

Sol: Being a polynomial function, f(x) is continuous on [1,3], differentiable on (1,3) and f(1)=f(3)=0.

$\therefore$ f(x) satisfies all the three conditions of Rolle's theorem.

Now, f(x)=(x–1)(x–2)(x–3)

$\Rightarrow f'(x) = (x-1)(x-2)(1)+(x-1)(x-3)(1)+(x-2)(x-3)(1) = 3x^2-12x+11.$

$\therefore$ By Rolle's theorem $\exists$ c $\in(1,3)$ such that $f'(c)=0 \Rightarrow 3c^2-12c+11=0$

$$\Rightarrow c = \frac{12 \pm \sqrt{144-132}}{2(3)} = \frac{12 \pm \sqrt{8}}{6} = \frac{12 \pm 2\sqrt{2}}{6} = 2 \pm \frac{1}{\sqrt{3}}$$

Both these values lie in the open interval (1,3) and these two roots are such that the derivatives vanish at these points.

---

14. Find the intervals in which the function $f(x) = \sin^4 x + \cos^4 x \; \forall x \in \left[0, \frac{\pi}{2}\right]$ is increasing and decreasing.

Sol: $f(x) = \sin^4x + \cos^4x = (\sin^2x)^2 + (\cos^2x)^2 = (\sin^2x + \cos^2x)^2 - 2\sin^2x\cos^2x = 1 - \frac{1}{2}\sin^2 2x$

$$f'(x) = \frac{-1}{2} 2\sin 2x.\cos 2x(2) = -2\sin 2x.\cos 2x = -\sin 4x$$

We know $\sin\theta > 0$ for $0 < \theta < \pi$. Now $0 < 4x < \pi \Rightarrow 0 < x < \frac{\pi}{4}$

Also $\sin\theta < 0$ for $\pi < \theta < 2\pi$. Now $\pi < 4x < 2\pi \Rightarrow \frac{\pi}{4} < x < \frac{\pi}{2}$

(i) f(x) is increasing when $f'(x) > 0 \Rightarrow -\sin 4x > 0 \Rightarrow \sin 4x < 0 \Rightarrow x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

(ii) f(x) is decreasing when $f'(x) < 0 \Rightarrow -\sin 4x < 0 \Rightarrow \sin 4x > 0 \Rightarrow x \in \left(0, \frac{\pi}{4}\right)$