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DIGITAL MATERIAL

FUNCTIONS-INDEX

1.FUNCTIONS

- Restricted Relations

1. INTRODUCTION PAGE

Sections	No. of periods (10 to 13)	Weightage in IPE [2x2]+[1x7]=11
0. Sets, Relations	1	
1. Basic Concepts of Functions	3 to 4	2 Marks
2. Theorems on Functions	4	7 Marks
3. Domain & Range of Real Functions	3 to 4	2 Marks

Basic concepts of functions are already covered in lower classes. We recapitulate some basic concepts of functions and extend them for further treatment.

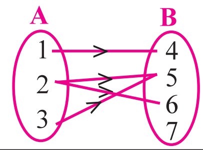
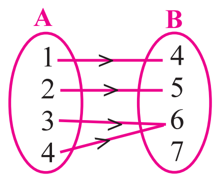
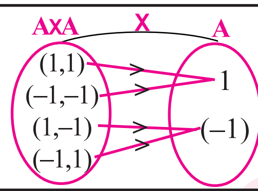
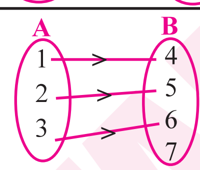
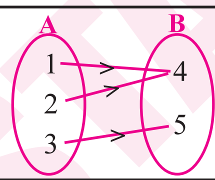
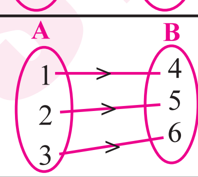
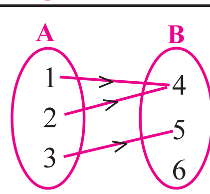
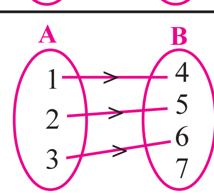
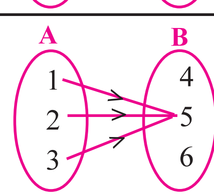
Function is basically a relation and it can be expressed by means of arrow diagrams, tables, set of ordered pairs, expressions, graphs, etc.

Section-1 consists of Definition of function, Equality of functions, Composition of functions, Algebra of functions, Types of functions - One one function, Onto function, Bijective function, Identity function, inverse of a function etc.,

Properties of certain types of functions in various aspects are dealt with in some important theorems, in section-2

The 'domain' of real function (with special emphasis), and the range of real function are discussed in section-3.

2 . THE TABLE OF FUNCTIONS

Relation/ Function from A to B	Example	No. of mappings when n(A)=m, n(B)=n	Remarks
1. **Relation**: A subset of AxB. Relation is a set of ordered pairs.		2^{mn}	None to none, one to one, one to many, many to one, many to many correspondences exist
2. **Function**: A relation from A to B such that each element of A is associated with only one element of B. A relation f:A→B is **well defined** if for $a_1,a_2 \in A,\ a_1=a_2 \Rightarrow f(a_1)=f(a_2)$		n^m No. of relations but not functions $=2^{mn}-n^m$	One to many correspondence does not exist. Function is a permutation with repetition.
3. **Binary operation**: A function from A x A to A		n^{n^2} Commutative binary operations $= n^{(n(n+1)/2)}$	+, x are Binary on N –, ÷ are not Binary on N
4. **One-One function:** A function f:A→B is called a one-one function if $f(x_1)=f(x_2) \Rightarrow x_1=x_2\ \forall x_1,x_2 \in A$		${}^nP_m$, $(m \le n)$	Different elements of A have different images. One-one function is a permutation without repetition.
5. **Onto function:** A function f:A→B is said to be an onto function ∀ y∈B ∃ x∈A such that f(x)=y. i.e., f(A)=B		2^m-2, (when n=2) $3^m-3.2^m+3$, (when n=3)	$n(A) \ge n(B)$ Range=codomain
6. **Bijective function:** A function f:A→B is said to be a bijective function if f is both one one and onto function.		$n!$	$n(A)=n(B)$ Inverse for f exists only when f is a bijective.
7. **Many-one function:** A function f:A→B is said to be a many one function if, two or more different elements of A have the same image in B.		$n^m - {}^nP_m$, $(m \le n)$ n^m, $(m>n)$	A many-one function can never be a one-one. A function which is neither one-one nor onto may be a many-one
8. **Into function:** A function f: A→B is called an into function if $f(A) \subset B$.		n^m, $(m<n)$	An into function can never be an onto function
9. **Constant function:** A function f: A→B is called a constant function, if the range of f consists of only one element		m	A constant function is onto if its codomain is singleton. A constant function one-one if its domain is singleton.

ADDITIONAL QUESTIONS WITH SOLUTIONS

1. If $f(x) = \frac{x-1}{x+1}$, $x \neq \pm 1$, show that $fof^{-1}(x)=x$.

Sol: Let $f(x) = y = \frac{x-1}{x+1} \Rightarrow y(x+1) = x-1 \Rightarrow yx+y = x-1 \Rightarrow yx-x = -1-y$.

$\Rightarrow x(y-1) = -1-y \Rightarrow x = \frac{1+y}{1-y} \Rightarrow f^{-1}(y) = \frac{1+y}{1-y} \Rightarrow f^{-1}(x) = \frac{1+x}{1-x}$

$\therefore (fof^{-1})(x) = f[f^{-1}(x)] = f\left(\frac{1+x}{1-x}\right) = \frac{\frac{1+x}{1-x}-1}{\frac{1+x}{1-x}+1} = \frac{1+x-(1-x)}{1+x+(1-x)} = \frac{2x}{2} = x$

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2. If $f:[1,\infty) \to [1,\infty)$ defined by $f(x)=2^{x(x-1)}$ then find $f^{-1}(x)$.

Sol: Let $2^{x(x-1)}=y$

$\Rightarrow \log_2 y = x(x-1)$ [$\because a^x = N \Rightarrow \log_a N = x$]

$\Rightarrow x^2 - x - \log_2 y = 0 \Rightarrow x = \frac{1 \pm \sqrt{1+4\log_2 y}}{2} \Rightarrow f^{-1}(y) = \frac{1+\sqrt{1+4\log_2 y}}{2}$ [$\because f^{-1}(y) > 1$]

$\Rightarrow f^{-1}(x) = \frac{1+\sqrt{1+4\log_2 x}}{2}$

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3. If $f(x) = e^x$ and $g(x) = \log_e x$, then show that fog = gof and find f^{-1} and g^{-1}.

Sol: $(fog)(x) = f(g(x)) = f(\log_e x) = e^{\log_e x} = x$

$(gof)(x) = g(f(x)) = g(e^x) = \log_e(e^x) = x\log_e e = x(1) = x$.

$\therefore (fog) = (gof) = x = I(x)$. Hence $f^{-1}(x) = g(x)$ and $g^{-1}(x) = f(x)$.

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4. Determine whether the function $f:(0,\infty) \to R$ defined by $f(x)=\log_e x$ is one one (or) onto (or) bijection.

Sol: (i)Let $x_1, x_2 \in (0,\infty)$ be such that $f(x_1) = f(x_2) \Rightarrow \log_e(x_1) = \log_e(x_2) \Rightarrow x_1 = x_2$

$\therefore$ f is one one.

(ii)We know the range of the logarithmic function is R.

Hence, the range of f = R = codomain of f $\therefore$ f is onto.

Thus, f is both one one and onto, hence bijective.

5. Find the inverse of (i) $f:R\to R$ such that $f(x)=2x+3$ (ii) $g:R^+\to R$ such that $g(x)=\log_5 x$.

Sol: (i) Let $f(x)=y\Rightarrow 2x+3=y\Rightarrow 2x=y-3\Rightarrow x=\frac{y-3}{2}$

$\Rightarrow f^{-1}(y)=\frac{y-3}{2}\ \therefore f^{-1}(x)=\frac{x-3}{2}$

(ii) Let $g(x)=y\Rightarrow \log_5 x=y\Rightarrow x=5^y\Rightarrow g^{-1}(y)=5^y\ \therefore g^{-1}(x)=5^x$

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6. If A={a,b,c}, B={0,1,2} then determine whether the following relations from A to B are one one or onto or bijective functions. (a) f={(a,1), (b,0), (c,2)} (b) g={(a,2), (b,0), (c,2)}

Sol: (a) f={(a,1), (b,0), (c,2)}. Here, all the elements in the domain A have unique images in the codomain B. So f is a function

(i) Different elements of the domain A have different images ∴ f is one one.

(ii) Range of f is f(A)={1,0,2} = B, the codomain of f ∴ f is onto

(b) g={(a,2), (b,0), (c,2)}. Clearly g is a function.

(i) Different elements a,c of the domain A have the same image 2 ∴ g is not one one

(ii) Range of g is g(A)={2,0}≠B, the codomain of g ∴ g is not onto

∴ g is neither one one nor onto.

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7. Determine whether the following functions are even or odd or neither even nor odd (i) $f:R\to R$ such that $f(x)=x^2$ (ii) $g:R\to R$ such that $g(x)=\sin x$ (iii) $h:R\to R$ such that $h(x)=x+x^2$

Sol: (i) Given $f(x)=x^2$

Now $f(-x)=(-x)^2=x^2=f(x)\Rightarrow f(-x)=f(x)$ ∴ f is an even function.

(ii) Given that $g:R\to R$ is such that $g(x)=\sin x$

Now, $g(-x)=\sin(-x)=-\sin x=-g(x)\Rightarrow g(-x)=-g(x)$ ∴ g is an odd function.

(iii) Given that $h:R\to R$ is such that $h(x)=x+x^2$

Now, $h(-x)=-x+(-x)^2=-x+x^2\neq\pm h(x)$ ∴ h is neither even nor odd

8. Let $f(x) = x^2$, $g(x) = 2^x$. Then solve the equation $(fog)(x)=(gof)(x)$ EAM Q

Sol: Given $f(x) = x^2$ and $g(x) = 2^x$.

Now $(fog)(x) = f(g(x)) = f(2^x) = (2^x)^2 = 2^{2x}$ ....(1)

Also, $(gof)(x) = g(f(x)) = g(x^2) = (2)^{x^2}$....(2)

From (1) & (2), $(fog)(x)=(gof)(x) \Rightarrow 2^{2x} = (2)^{x^2} \Rightarrow 2x=x^2 \Rightarrow x^2-2x=0$

$\Rightarrow x(x-2)=0$ $\therefore$ The solution of the given equation is x=0, 2.

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9. If f, g:R→R are defined $f(x) = \begin{cases} 0 \text{ if } x \in Q \\ 1 \text{ if } x \notin Q \end{cases}$ and $g(x) = \begin{cases} -1 \text{ if } x \in Q \\ 0 \text{ if } x \notin Q \end{cases}$ then find $(fog)(\pi)+(gof)(e)$.

Sol: $(fog)(\pi) = f(g(\pi)) = f(0) = 0$ and

$(gof)(e) = g(f(e)) = g(1) = -1$ $\therefore (fog)(\pi) + (gof)(e) = 0-1 = -1$

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10. Find the domain of definition of the function y(x), given by the equation $2^x+2^y=2$.

Sol: Given that $2^x+2^y=2 \Rightarrow 2^x = 2-2^y < 2$ $(\because 2^y>0) \Rightarrow 2^x < 2$

$\Rightarrow \log_2 2^x < \log_2 2 \Rightarrow x(\log_2 2) < \log_2 2 \Rightarrow x<1$ $\therefore$ Domain $= (-\infty, 1)$

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11. Find the range of $\log|4-x^2|$

Sol: We know that for x>0, Range of log x is R.

We have $|4-x^2|>0, \forall x \in R-\{-2,2\}$. Hence range of $\log|4-x^2|$ is R.

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12. Find the range of $\sqrt{[x]-x}$

Sol: $\sqrt{[x]-x}$ is defined when $[x]-x \geq 0$.

We know for any $x \in Z$, we have $[x]-x=0$ $\therefore [x]-x \geq 0$ holds true $\forall x \in Z$

$\therefore$ Range of f(x) is {0}

13. Find the range of $\frac{\sin\pi[x]}{1+[x]^2}$

Sol: We know that $\forall x\in R,[x]\in Z$. $[\because (x-1)<[x]\le x]$

Now, for $[x]\in Z$ we have Nr. $\sin \pi[x]=0$. $[\because \sin([x]\pi)=\sin n\pi=0,\ \forall n=[x]\in Z]$

Also Dr. $1+[x]^2>0,\ \forall x\in R$

$\therefore \forall x\in R,\ f(x)=0$ $\therefore$ Range of f(x) is $\{0\}$

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14. Find the range of $\sqrt{9+x^2}$

Sol: We know that $\forall x\in R, 0\le x^2<\infty \Rightarrow 9\le 9+x^2<\infty \Rightarrow 3\le \sqrt{9+x^2}<\infty$

$\therefore$ Range of $\sqrt{9+x^2}$ is $[3,\infty)$

(or) At x=0 we get the minimum value $\sqrt{9+0}=\sqrt{9}=3$.

Hence Range is $[3,\infty)$

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15. Prove that the real valued function $f(x)=\frac{x}{e^x-1}+\frac{x}{2}+1$ is an even function on $R-\{0\}$

Sol: Given $f(x)=\frac{x}{e^x-1}+\frac{x}{2}+1$

Now, $f(-x)=\frac{-x}{e^{-x}-1}+\frac{(-x)}{2}+1$

$=\frac{-x}{\left(\frac{1}{e^x}\right)-1}-\frac{x}{2}+1=\frac{-x(e^x)}{1-e^x}-\frac{x}{2}+1$

EAM Q

$=\frac{x(e^x)}{e^x-1}-\frac{x}{2}+1=\frac{x(e^x)}{e^x-1}-x+\frac{x}{2}+1$ $\left[\because \text{we write } \frac{-x}{2}=-x+\frac{x}{2}\right]$

$=\frac{xe^x-x(e^x-1)}{e^x-1}+\frac{x}{2}+1=\frac{\cancel{xe^x}-\cancel{xe^x}+x}{e^x-1}+\frac{x}{2}+1=\frac{x}{e^x-1}+\frac{x}{2}+1=f(x)$

$\therefore f(-x)=f(x)\Rightarrow f(x)$ is an even function.

16. If the function f:R→R defined by $f(x)=\dfrac{3^x+3^{-x}}{2}$ then S.T f(x+y)+f(x–y)=2f(x)f(y)

Sol: Given that $f(x)=\dfrac{3^x+3^{-x}}{2}$.

Now, $f(x+y)=\dfrac{3^{x+y}+3^{-(x+y)}}{2}=\dfrac{3^x.3^y+3^{-x}.3^{-y}}{2}$ .....(1)

Also, $f(x-y)=\dfrac{3^{x-y}+3^{-(x-y)}}{2}=\dfrac{3^x.3^{-y}+3^{-x}.3^y}{2}$ ....(2)

L.H.S = f(x+y)+f(x–y)

$=\dfrac{(3^x.3^y+3^{-x}.3^{-y})+(3^x.3^{-y}+3^{-x}.3^y)}{2}=\dfrac{1}{2}[3^x(3^y+3^{-y})+3^{-x}(3^{-y}+3^y)]$

$=\dfrac{1}{2}(3^x+3^{-x})(3^y+3^{-y})=2\left[\left(\dfrac{3^x+3^{-x}}{2}\right)\left(\dfrac{3^y+3^{-y}}{2}\right)\right]=2f(x).f(y)$ =R.H.S

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17. If the function f:R→R defined by $f(x)=\dfrac{4^x}{4^x+2}$, then Show that f(1–x)=1–f(x), and hence deduce the value of $f\left(\dfrac{1}{4}\right)+2f\left(\dfrac{1}{2}\right)+f\left(\dfrac{3}{4}\right)$

EAM Q

Sol: (i) Given that, $f(x)=\dfrac{4^x}{4^x+2}$. To claim the result we show that f(x)+f(1–x)=1

Now, $f(x)+f(1-x)=\dfrac{4^x}{4^x+2}+\dfrac{4^{1-x}}{4^{1-x}+2}=\dfrac{4^x}{4^x+2}+\dfrac{4.4^{-x}}{4.4^{-x}+2(1)}=\dfrac{4^x}{4^x+2}+\dfrac{\cancel{4.4^{-x}}}{\cancel{2.4^{-x}}(2+4^x)}$

$=\dfrac{4^x}{4^x+2}+\dfrac{2}{2+4^x}=\dfrac{\cancel{4^x+2}}{\cancel{4^x+2}}=1$ $\Rightarrow$f(x)+f(1–x)=1 . Hence follows the result

(ii) $f\left(\dfrac{1}{4}\right)+2f\left(\dfrac{1}{2}\right)+f\left(\dfrac{3}{4}\right)=f\left(\dfrac{1}{4}\right)+f\left(\dfrac{3}{4}\right)+2f\left(\dfrac{1}{2}\right)=\left(f\left(\dfrac{1}{4}\right)+f\left(\dfrac{3}{4}\right)\right)+\left(f\left(\dfrac{1}{2}\right)+f\left(\dfrac{1}{2}\right)\right)$

$=\left(f\left(\dfrac{1}{4}\right)+f\left(1-\dfrac{1}{4}\right)\right)+\left(f\left(\dfrac{1}{2}\right)+f\left(1-\dfrac{1}{2}\right)\right)=1+1=2$ $[\because f(x)+f(1-x)=1]$

18. If f(x)=cos(logx), then show that $f\left(\frac{1}{x}\right)f\left(\frac{1}{y}\right)-\frac{1}{2}\left[f\left(\frac{x}{y}\right)+f(xy)\right]=0$

Sol: Given $f(x)=\cos(\log x)$

$\Rightarrow f\left(\frac{1}{x}\right)=\cos\left(\log\left(\frac{1}{x}\right)\right)=\cos(\log 1-\log x)=\cos(-\log x)=\cos(\log x)$ [$\because \log 1=0, \cos(-\theta)=\cos\theta$]

Similarly, $f\left(\frac{1}{y}\right)=\cos(\log y)$

Also, $f\left(\frac{x}{y}\right)=\cos\log\left(\frac{x}{y}\right)=\cos(\log x-\log y)$ and $f(xy)=\cos\log(xy)=\cos(\log x+\log y)$

Hence, $f\left(\frac{x}{y}\right)+f(xy)=\cos(\log x-\log y)+\cos(\log x+\log y)=2\cos(\log x)\cos(\log y)$

[$\because \cos(A-B)+\cos(A+B)=2\cos A.\cos B$]

Now, $LHS=f\left(\frac{1}{x}\right)f\left(\frac{1}{y}\right)-\frac{1}{2}\left[f\left(\frac{x}{y}\right)+f(xy)\right]$

$=\cos(\log x)\cos(\log y)-\frac{1}{\not 2}[\not 2\cos(\log x)\cos(\log x)]=0=R.H.S$