1. COMPLEX NUMBERS

Sections	No. of periods (6)	Weightage in IPE [1x2+1x4=6]
1. Complex Numbers-I	3	2 or 4 marks
2. Complex Numbers-II	3	2 or 4 marks

The set of real numbers R is extended to the set of complex numbers C, inorder to solve the equations like $x^2+1=0$. *As a way out, a new number whose square is –1, is introduced and it is denoted by i. Thus* $i^2=-1$ *and i is the square root of –1. Numbers of the form i, 2i, 3i, i/2, –3/2i, etc., are called imaginary numbers.*

The real line (one dimension) is completely filled by the set of all reals and hence there is no place for imaginary numbers. Hence we are forced to enter into the second dimension. Here, all the imaginary numbers are placed on a line perpendicular to the real axis, called imaginary axis. Now, the Cartesian product of these two sets of numbers, yield the so-called complex numbers. A complex number is written as x+iy, denoted by z, where x,y are real numbers. Here, we have to note that, this positive sign + does not denote the addition as we understood hither. It is just conventionally used to reveal the fact that the real number x is associated with the imaginary number yi. A complex number is also represented by ordered pair of reals.

Thus, $C=\{(a,b),a,b\in R\} = R\times R$

The plane depicting every point by means of a complex number is called Argand plane. This Argand plane just corresponds to the Cartesian plane in most of the aspects. In this regard, the role of a complex number z=x+iy is compensated by the point P(x,y). The point P(x,y) is called the image of the complex number z=x+iy and z is said to be the affix or complex coordinate of the point P. The complex number z=x+iy representing the point P(x,y) is also represented by the position vector OP.

In this topic, we have discussed the fundamental operations on complex numbers, structural properties of complex numbers, square root of complex numbers, cube roots and 4^{th} *roots of unity, conjugate of a complex number, modulus of a complex number, argument of a complex number, polar form of a complex number etc.,*

Demoivre's theorem, one of the important applications of the topic "complex numbers", with the use of which we (i) solve some of the algebraic equations with higher powers (ii) find expansions for $\sin n\theta$, $\cos n\theta$ *etc.*

SYNOPSIS POINTS

1. Complex Numbers are expressed in the form z=x+iy or (x,y) where $x,y\in R$, $i=\sqrt{-1}$

2.1. If z=x+iy then x is called the Real part of z and it is denoted by Re(z)

and y is called the Imaginary part of z and it is denoted by Im(z)

2.2. A complex number is said to be (i) purely real if its imaginary part is zero.

(ii) purely imaginary if its real part is zero.

3. **Equality of complex numbers**: Two complex numbers $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$ are equal

i.e., $x_1+iy_1=x_2+iy_2 \Leftrightarrow x_1=x_2$ and $y_1=y_2$

i.e., corresponding real parts are equal and corresponding imaginary parts are equal.

4. If $z_1=x_1+iy_1$, $z_2=x_2+iy_2$ then (i) $z_1+z_2=(x_1+x_2)+i(y_1+y_2)$ (ii) $z_1-z_2=(x_1-x_2)+i(y_1-y_2)$

(iii) $z_1.z_2=(x_1+iy_1)(x_2+iy_2)=(x_1x_2-y_1y_2)+i(y_1x_2+x_1y_2)$

(iv) $\frac{z_1}{z_2}=\frac{x_1+iy_1}{x_2+iy_2}=\frac{(x_1+iy_1)(x_2-iy_2)}{(x_2+iy_2)(x_2-iy_2)}=\left(\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}\right)+i\left(\frac{y_1x_2-x_1y_2}{x_2^2+y_2^2}\right)$

Note : (i) $\sqrt{-a}=i\sqrt{a}, a\in R^+$ (ii) $(x+iy)^2=(x^2-y^2)+2ixy$ (iii) $(x+iy)(x-iy)=x^2+y^2$ (iv) $\frac{1+i}{1-i}=i$

5. If z=x+iy then its (i) conjugate is $\bar{z}=x-iy$ (ii) additive inverse is $-z=-x-iy$

(iii) multiplicative inverse is $\frac{x}{x^2+y^2}-\frac{iy}{x^2+y^2}$ (iv) modulus is $|z|=\sqrt{x^2+y^2}$ (v) $\text{Arg } z=\text{Tan}^{-1}\left(\frac{y}{x}\right)$

6. The square root of $a+ib$ is $\sqrt{a+ib}=\pm\left(\sqrt{\frac{r+a}{2}}+i\sqrt{\frac{r-a}{2}}\right)$, where $r=\sqrt{a^2+b^2}$

7.1.The roots of $x^3=1$ are called cube roots of unity, which are 1; $\omega=\frac{-1+i\sqrt{3}}{2}$; $\omega^2=\frac{-1-i\sqrt{3}}{2}$

7.2. $1+\omega+\omega^2=0$, hence we have $1+\omega=-\omega^2$, $1+\omega^2=-\omega$, $\omega+\omega^2=-1$

8. The roots of $x^4=1$ are called fourth roots of unity, which are 1,–1, i,–i

9. **Mod amp.form or polar form**:

If $z=x+iy$ then $z=r(\cos\theta+i\sin\theta)$ is called Mod-amplitude form of x+iy where

(i) r is the modulus of z, i.e., $r=\sqrt{x^2+y^2}$

(ii) The principal amplitude $\text{Tan}^{-1}\left(\frac{y}{x}\right)=\theta\in(-\pi,\pi]$ can be determined

according the quadrant in which the point represented by (x,y) lies.

10.1. If Arg $z=\theta$ then $\text{Arg }\bar{z}=-\theta$

10.2. If Arg $z_1=\theta_1$ and Arg $z_2=\theta_2$ then (i) Arg $(z_1.z_2)=\theta_1+\theta_2$ (ii) $\text{Arg}(z_1/z_2)=\theta_1-\theta_2$

11. $\cos\theta+i\sin\theta$ is simply written as $\text{cis}\theta$; $\text{cis}\theta=e^{i\theta}$; $\text{cis}\theta_1.\text{cis}\theta_2=\text{cis}(\theta_1+\theta_2)$; $\frac{\text{cis}\theta_1}{\text{cis}\theta_2}=\text{cis}(\theta_1-\theta_2)$

ADDITIONAL QUESTIONS WITH SOLUTIONS

1 **If z = x+iy and if the point P in the Argand plane represents z, then describe geometrically the locus of z, satisfying the equation 2|z–2| =|z–1|**

Sol: Given $z = x+iy \Rightarrow 2|z-2|=|z-1| \Rightarrow 2|x+iy-2| = |x+iy-1| \Rightarrow 2|(x-2)+iy| =|(x-1)+iy|$

$\Rightarrow 2\sqrt{(x-2)^2+y^2} = \sqrt{(x-1)^2+y^2} \Rightarrow 4[(x-2)^2+y^2] = (x-1)^2+y^2$

$\Rightarrow 4[(x^2-4x+4)+y^2] = (x^2-2x+1)+y^2 \Rightarrow 3x^2+3y^2-14x+15=0$

$\therefore$ This locus represents the equation of a circle.

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2 **If z=x+iy and if the point P in the Argand plane represents z, then describe geometrically the locus of P satisfying the equations.**

(i) |2z–3|=7 (ii) $|z|^2$=4Re(z+2) (iii) $|z+i|^2-|z-i|^2=2$ (iv) |z+4i|+|z–4i|=10

Sol: i) Given $z=x+iy \Rightarrow |2z-3| =7 \Rightarrow |2(x+iy)-3|=7 \Rightarrow |(2x-3)+i(2y)|=7$

$\Rightarrow \sqrt{(2x-3)^2+4y^2} = 7 \Rightarrow (2x-3)^2+4y^2 = 49 \Rightarrow (4x^2+9-12x)+4y^2 = 49$

$\Rightarrow 4x^2+4y^2-12x-40=0 \Rightarrow x^2+y^2-3x-10=0$

ii) Given $z=x+iy \Rightarrow |z| = \sqrt{x^2+y^2} \Rightarrow |z|^2 = x^2+y^2$

Now, z+2=(x+iy)+2=(x+2)+iy. Hence Re(z+2)=x+2

$\therefore |z|^2 = 4\,\mathrm{Re}(z+2) \Rightarrow x^2+y^2 = 4(x+2) \Rightarrow x^2+y^2-4x-8=0$

iii) Given $z = x+iy \Rightarrow |z+i|^2-|z-i|^2 = 2 \Rightarrow |x+iy+i|^2-|x+iy-i|^2 = 2$

$\Rightarrow |x+i(y+1)|^2-|x+i(y-1)|^2 = 2 \Rightarrow (\sqrt{x^2+(y+1)^2})^2-(\sqrt{x^2+(y-1)^2})^2 = 2$

$\Rightarrow \cancel{x^2}+(y+1)^2-\cancel{x^2}-(y-1)^2 = 2 \Rightarrow 4(y)(1) = 2 \Rightarrow 2y = 1$ $[\because (a+b)^2-(a-b)^2 = 4ab]$

iv) Given $z = x+iy \Rightarrow |z+4i|+|z-4i| = 10\ |x+iy+4i|+|x+iy-4i| = 10$

$\Rightarrow |x+i(y+4)|+|x+i(y-4)| = 10 \Rightarrow \sqrt{x^2+(y+4)^2}+\sqrt{x^2+(y-4)^2} = 10$

$\Rightarrow \sqrt{x^2+(y+4)^2} = 10-\sqrt{x^2+(y-4)^2}$

Squaring on both sides, we get $x^2+(y+4)^2 = 100+x^2+(y-4)^2-20\sqrt{x^2+(y-4)^2}$

$\Rightarrow (y+4)^2-(y-4)^2 = 100-20\sqrt{x^2+(y-4)^2} \Rightarrow 4(y)(4) = 100-20\sqrt{x^2+(y-4)^2}$

$\Rightarrow 4y-25 = -5\sqrt{x^2+(y-4)^2}$.

Again squaring on both sides, $16y^2-200y+625 = 25[x^2+(y-4)^2]$

$= 25(x^2+y^2-8y+16) = 25x^2+25y^2-200y+400 \Rightarrow 25x^2+9y^2 = 225$

3 **If $\frac{z_2}{z_1},(z_1 \neq 0)$, is an imaginary number then find the value of $\left|\frac{2z_1+z_2}{2z_1-z_2}\right|$**

Sol: Given that $\frac{z_2}{z_1},(z_1 \neq 0)$ is an imaginary number . So we take $\frac{z_2}{z_1}=0+ki=ki$

$$\therefore \left|\frac{2z_1+z_2}{2z_1-z_2}\right| = \left|\frac{\frac{2z_1}{z_1}+\frac{z_2}{z_1}}{\frac{2z_1}{z_1}-\frac{z_2}{z_1}}\right| = \left|\frac{2+\frac{z_2}{z_1}}{2-\frac{z_2}{z_1}}\right| = \left|\frac{2+ik}{2-ik}\right| = \frac{|2+ik|}{|2-ik|} = \frac{\sqrt{4+k^2}}{\sqrt{4+k^2}} = 1$$

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4 **If $\frac{z_3-z_1}{z_2-z_1}$ is a real number, show that the points represented by the complex numbers z_1, z_2, z_3 are collinear.**

Sol: Let $\frac{z_3-z_1}{z_2-z_1}=k,R \Rightarrow z_3-z_1=k(z_2-z_1)=kz_2-kz_1 \Rightarrow kz_1-z_1=kz_2-z_3 \Rightarrow z_1(k-1)=kz_2-z_3$

$\Rightarrow z_1=\frac{kz_2-z_3}{k-1}$. This is the 'section formula' for external division.

Thus z_1 represents a collinear point which divides the line joining the points z_2, z_3 in the ratio k:1 externally. $\therefore$ z_1, z_2, z_3 are collinear

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5 **Show that the points in the Argand diagram represented by the complex numbers z_1,z_2,z_3 are collinear, if and only if there exists three real numbers p,q,r not all zero, satisfying $pz_1+qz_2+rz_3=0$ and p+q+r=0**

Sol: z_1, z_2, z_3 are collinear $\Leftrightarrow$ z_3 lies on the line joining z_1 and z_2.

$\Leftrightarrow z_3=\frac{mz_2+nz_1}{m+n} \Leftrightarrow (m+n)z_3=mz_2+nz_1 \Leftrightarrow nz_1+mz_2-(m+n)z_3=0$

$\Leftrightarrow pz_1+qz_2+rz_3=0$ where p=n, q=m, r = –m–n such that p+q+r=0

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6 **If z = x +iy and the point P represent z in the Argand plane and $\left|\frac{z-a}{z+\bar{a}}\right|=1$, Re (a) ≠ 0, then find the locus of P.** TS 15

Sol: Given z = x + iy ; Let a = α + iβ ⇒ $\bar{a}$ = α–iβ

Now $\left|\frac{z-a}{z+\bar{a}}\right|=1 \Rightarrow \frac{|z-a|}{|z+\bar{a}|}=1 \Rightarrow |z-a|=|z+\bar{a}|$

$\Rightarrow |x+iy-(\alpha+i\beta)|=|x+iy+(\alpha-i\beta)| \quad \Rightarrow |(x-\alpha)+i(y-\beta)|=|(x+\alpha)+i(y-\beta)|$

$\Rightarrow \sqrt{(x-\alpha)^2+(y-\beta)^2}=\sqrt{(x+\alpha)^2+(y-\beta)^2}$

By squaring on both sides, we get ($(x-\alpha)^2+\cancel{(y-\beta)^2}=(x+\alpha)^2+\cancel{(y-\beta)^2}$

$\Rightarrow (x+\alpha)^2-(x-\alpha)^2=0$

$\Rightarrow 4\alpha x=0$ [$\because$ Re (a) ≠ 0 ⇒ α ≠ 0]

$\Rightarrow x=0$

7 **The points P,Q denote the complex numbers z_1,z_2 in the Argand diagram. O is the origin. If $z_1\bar{z}_2+\bar{z}_1z_2=0$ then show that $\angle POQ=90^\circ$.**

Sol: Let $P=(x_1,y_1)\Rightarrow z_1=x_1+iy_1$;$Q=(x_2,y_2)\Rightarrow z_2=x_2+iy_2$ and $O=(0,0)$

$z_1=x_1+iy_1\Rightarrow\bar{z}_1=x_1-iy_1$ and $z_2=x_2+iy_2\Rightarrow\bar{z}_2=x_2-iy_2$

Given that $z_1\bar{z}_2+\bar{z}_1z_2=0\Rightarrow(x_1+iy_1)(x_2-iy_2)+(x_1-iy_1)(x_2+iy_2)=0$

$\Rightarrow(x_1x_2-\cancel{ix_1y_2}+\cancel{ix_2y_1}+y_1y_2)+(x_1x_2+\cancel{ix_1y_2}-\cancel{ix_2y_1}+y_1y_2)=0\Rightarrow 2(x_1x_2+y_1y_2)=0$

$\Rightarrow x_1x_2+y_1y_2=0\Rightarrow x_1x_2=-y_1y_2\Rightarrow\left(\frac{-y_1}{x_1}\right)\left(\frac{-y_2}{x_2}\right)=-1$

$\Rightarrow(\text{Slope of }\overleftrightarrow{OP})(\text{Slope of }\overleftrightarrow{OQ})=-1\Rightarrow\angle POQ=90^\circ$

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8 **If z_1,z_2 are two non-zero complex numbers satisfying $|z_1+z_2|=|z_1|+|z_2|$ show that $\text{Arg } z_1-\text{Arg } z_2=0$.**

Sol: Let $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$ $\therefore z_1+z_2=(x_1+iy_1)+(x_2+iy_2)=(x_1+x_2)+i(y_1+y_2)$

$\Rightarrow|z_1+z_2|=\sqrt{(x_1+x_2)^2+(y_1+y_2)^2}$

Given that $|z_1+z_2|=|z_1|+|z_2|$ $\Rightarrow\sqrt{(x_1+x_2)^2+(y_1+y_2)^2}=\sqrt{x_1^2+y_1^2}+\sqrt{x_2^2+y_2^2}$

Squaring on both sides

$(x_1+x_2)^2+(y_1+y_2)^2=x_1^2+y_1^2+x_2^2+y_2^2+2\sqrt{\left(x_1^2+y_1^2\right)\left(x_2^2+y_2^2\right)}$

$\Rightarrow(x_1^2+x_2^2+2x_1x_2)+(y_1^2+y_2^2+2y_1y_2)=x_1^2+x_2^2+y_1^2+y_2^2+2\sqrt{\left(x_1^2+y_1^2\right)\left(x_2^2+y_2^2\right)}$

$\Rightarrow 2x_1x_2+2y_1y_2=2\sqrt{\left(x_1^2+y_1^2\right)\left(x_2^2+y_2^2\right)}$

$\Rightarrow x_1x_2+y_1y_2=\sqrt{\left(x_1^2+y_1^2\right)\left(x_2^2+y_2^2\right)}$

Squaring again, $(x_1x_2+y_1y_2)^2=(x_1^2+y_1^2)(x_2^2+y_2^2)$

$\Rightarrow x_1^2x_2^2+y_1^2y_2^2+2x_1x_2y_1y_2=x_1^2x_2^2+x_1^2y_2^2+x_2^2y_1^2+y_1^2y_2^2$

$\Rightarrow x_1^2y_2^2+x_2^2y_1^2-2x_1x_2y_1y_2=0$

$\Rightarrow(x_1y_2-x_2y_1)^2=0\Rightarrow x_1y_2=x_2y_1\Rightarrow\frac{y_1}{x_1}=\frac{y_2}{x_2}\Rightarrow\text{Tan}^{-1}\left(\frac{y_1}{x_1}\right)=\text{Tan}^{-1}\left(\frac{y_2}{x_2}\right)$

$\Rightarrow\text{Arg }(z_1)=\text{Arg }(z_2)\Rightarrow\text{Arg }(z_1)-\text{Arg }(z_2)=0$

9 **Show that the equation of any circle in the complex plane is of the form $z\bar{z}+b\bar{z}+\bar{b}z+c=0\ (b\in C, c\in R)$**

Sol: We know that the general equation of circle is $x^2+y^2+2gx+2fy+c=0$, g,f,c $\in R$

Let z=x+iy then $\bar{z}=x-iy$. Since, $b\in C$ we take b=p+iq then $\bar{b}=p-iq$, p,q $\in R$

$\therefore\ z\bar{z}+b\bar{z}+\bar{b}z+c=0 \Rightarrow (x+iy)(x-iy)+(p+iq)(x-iy)+(p-iq)(x+iy)+c=0$

$\Rightarrow x^2+y^2+px+qy+iqx-ipy+px+qy+piy-qxi+c=0$

$\Rightarrow x^2+y^2+2px+2qy+c=0$, which is in the form $x^2+y^2+2gx+2fy+c=0$

Hence, the given equation represents the equation of a circle.

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10 **Find the eccentricity of the ellipse whose equation is $|z-4|+\left|z-\frac{12}{5}\right|=10$**

Sol: Given equation $|z-4|+\left|z-\frac{12}{5}\right|=10$

The equation of ellipse with foci S & S' and latusrectum 4a is SP + S'P = 2a

$\therefore S=(4,0)$, $S'\left(\frac{12}{5},0\right)$ and $2a=10 \Rightarrow a=5$

We know that the distance between the foci is $SS'=2ae$

$\Rightarrow 2ae=\left|4-\frac{12}{5}\right| \Rightarrow 2(5)e=\frac{8}{5} \Rightarrow e=\frac{4}{5}$

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11 **If the complex number z with argument θ, $0<\theta<\frac{\pi}{2}$ is such that |z–3i| = 3 then prove that $\left(\cot\theta-\frac{6}{z}\right)=i$**

Sol: Given that $0<\theta<\frac{\pi}{2}$. Then as x>0, y>0, $\tan\theta=\frac{y}{x} \Rightarrow \cot\theta=\frac{x}{y}$

Also z=x+iy $\Rightarrow \bar{z}=x-iy$. Then $z\bar{z}=(x+iy)(x-iy)=x^2+y^2$

Now, $|z-3i|=3 \Rightarrow |x+iy-3i|=|x+(y-3)i|=3 \Rightarrow \sqrt{x^2+(y-3)^2}=3$

$\Rightarrow x^2+(y-3)^2=9 \Rightarrow x^2+y^2=6y$ $\therefore z\bar{z}=6y$

Now $\cot\theta-\frac{6}{z}=\cot\theta-\frac{6(\bar{z})}{z(\bar{z})}=\cot\theta-\frac{6\bar{z}}{z\bar{z}}=\frac{x}{y}-\frac{6(x-iy)}{6y}=\frac{x}{y}-\frac{x}{y}+i=i.$